Textbook

Analysis 1 and 2

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$$\int_a^b f(x)\,dx = F(b) - F(a), \quad \frac{d}{dx}F(x) = f(x)$$

This version is not final and may be updated.

Academic Year: 2024/2025

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INTRODUCTION

The difference in math level between high school and university is in how deep, abstract, and rigorous it gets. In high school, math mainly focuses on calculations, using formulas, and solving problems within clear rules. Ideas like calculus and algebra are taught with an emphasis on intuition and practical use. At the university level, math becomes more abstract. Students need to work with formal definitions, theorems, and proofs. The transition isn't just about learning harder topics but also about changing how you think—focusing more on logic, precision, and general ideas instead of just memorizing steps.

This textbook gives a detailed and strict introduction to important ideas in calculus, mathematical analysis, and differential equations. It mixes theory with real-world applications, giving students the tools they need to solve problems in math, physics, engineering, and other fields. The chapters are organized to build step by step, helping students move smoothly from basic ideas to harder topics.

It acts as a bridge between basic calculus and advanced math studies. The textbook stresses clarity, rigor, and real-world usefulness. It's made for students who want a full understanding of calculus and its uses. By working through the material, readers will develop the thinking skills needed to handle many kinds of math problems. Whether you're focused on research, engineering, or modeling in science, this textbook lays the groundwork for success.

I hope it sparks curiosity and helps you appreciate the beauty and power of math.

It is crucial to understand that mathematics presented in English can differ slightly from how it is taught in other languages. For instance, certain concepts do not have direct equivalents or are defined differently. As an example,

there is no standard notion of "adjacent sequences" in English—I personally defined this term myself. Similarly, topics like "Limited Development," which form an important and well-defined chapter in other languages, are not commonly found in English mathematical literature. Additionally, there are some set symbols which are diffrent from a language to another, for example $\mathbb{N}$ denotes natural numbers without zero in english, but not in french, the natural numbers with zero are called whole numbers. Some one needs to proceed carefuly.

The content outlines key ideas in math, starting with a close look at the real number system. It explains its structure, properties, and the axioms that define it. Topics include algebraic and order axioms, the Archimedean property, the completeness axiom, and the topology of the real line. These ideas set the stage for understanding limits, continuity, and convergence.

Next, complex numbers are covered, including their algebraic properties, how they're represented in the complex plane, and their polar forms. This section connects real and complex math, offering tools for problems involving waves, oscillations, and similar topics in engineering and physics.

The study of sequences introduces important ideas like boundedness, convergence, monotonicity, and Cauchy sequences. These concepts help explain how functions and series behave and are key to proving results like the Cauchy Criterion and the Stolz-Cesaro Theorem.

The material then moves to real functions, focusing on limits, continuity, and differentiability. Important theorems about continuous and differentiable functions are explained, along with types of discontinuities and uniform continuity. Convex functions are also discussed, showing their role in optimization and economics.

The section on elementary functions gives a thorough look at logarithmic, exponential, trigonometric, and hyperbolic functions. These functions are studied from different angles, showing how they connect and why they're important for describing natural events.

The chapter on Taylor polynomials, little o and big O notation, and approximations near a point explores how functions behave locally and asymptotically. This is crucial for numerical methods and applied math.

Later sections cover integration, starting with indefinite integrals and moving to definite integrals using the Riemann integral. Techniques like integration by parts, substitution, and methods for rational, irrational, and trigonometric functions are explained in detail. Key results like the Fundamental Theorems of Calculus and inequalities like Hermite-Hadamard show the link between differentiation and integration.

Finally, the material touches on improper integrals, first-order differential equations, and second-order linear differential equations with constant coefficients. These topics are essential for solving real-world problems in physics, engineering, and economics, where systems often follow differential laws.

CHAPTER

1

REAL NUMBER SYSTEM

1.1 Algebraic and Order axioms

The real number system consists of the real numbers, together with the two operations, addition (denoted by $+$) and multiplication (denoted by $\times$) and the less than relation (denoted by $<$). One also singles out two particular real numbers, zero or 0 and one or 1. If a and b are real numbers, then so are $a+b$ and $a \times b$. We say that the real numbers are closed under addition and multiplication. We usually write

$$ab \text{ for } a \times b.$$

For any two real numbers a and b, the statement $a < b$ is either true or false. We will soon see that one can define subtraction and division in terms of $+$ and $\times$; and $\leq, >$, etc. can be defined from $<$. There are three categories of properties of the real number system: the algebraic properties, the order properties and the completeness property. We will discuss the completeness property in a later section of this chapter. Here we begin with certain basic algebraic and order properties, usually called the algebraic and order axioms, from which we can prove all the other algebraic and order properties of the real numbers. For all real numbers a, b and c :

1. $(a+b)+c = a+(b+c)$ (associative axiom for addition)

2. $a + 0 = 0 + a = a$ (additive identity axiom)
3. there is a real number, denoted $-a$, such that $a + (-a) = (-a) + a = 0$ (additive inverse axiom)
4. $a + b = b + a$ (commutative axiom for addition)
5. $(a \times b) \times c = a \times (b \times c)$ (associative axiom for multiplication)
6. $a \times 1 = 1 \times a = a$, moreover $0 \neq 1$ (multiplicative identity axiom)
7. if $a \neq 0$ then there is a real number, denoted a^{-1}, such that $a \times a^{-1} = a^{-1} \times a = 1$ (multiplicative inverse axiom)
8. $a \times (b + c) = a \times b + a \times c$ (distributive axiom)
9. $a \times b = b \times a$ (commutative axiom for multiplication)
10. exactly one of the following holds: $a < b$, $a = b$ or $b < a$ (trichotomy axiom)
11. if $a < b$ and $b < c$, then $a < c$ (transitivity axiom)
12. if $a < b$ then $a + c < b + c$ (addition and order axiom)
13. if $a < b$ and $0 < c$, then $a \times c < b \times c$ (multiplication and order axiom)

1.1.1 Remarks

- For equality, denoted by the symbol "=", we mean "the same thing as", or equivalently, "is the same real number as". We take "=" to be a logical notion and do not write axioms for it. [2] Instead, we use any property of "=" which follow from its logical meaning. For example $a = a$; if $a = b$ then $b = a$; if $a = b$ and $b = c$ then $a = c$; if $a = b$ and something is true of a then it's also true of b (since a and b denote the same real number!).
- When we write $a \neq b$, we just mean that a is different real number as b.
- The assertion $0 \neq 1$ in axiom 6 may seem silly. But it doesn't follow from the other axioms, since all the other axioms hold for the set containing just the number 0 .
- Some of the axioms are redundant. For example, from axiom 4 and the property $a + 0 = a$ it follows that $0 + a = a$. Similar comments apply to axiom 3 ; and because of axiom 6 to axioms 8 and 9.

1.1.2 Algebraic Structures

- Axioms $1, 2, 3, 4$ allow $\mathbb{R}$ the structure of an additive Abelian group.
- Axioms $5, 6, 7$ allow $\mathbb{R}$ the structure of a multiplicative group.
- Axioms $1, ..., 8$ allow us to justify that $\mathbb{R}$ is a field.
- Commutativity of the multiplication operation $\times$ makes $\mathbb{R}$ a commutative field.

From axiom 8, we can write

$$a\left(b+(-b)\right)=0=ab+a\left(-b\right),$$

which means

$$-\left(ab\right)=a\left(-b\right)$$

same as above we obtain

$$-\left(ab\right)=\left(-a\right)b.$$

1.1.3 Algebraic consequences

Certain not so obvious "rules", such as "the product of minus times minus is plus" and the rule for adding two fractions, follow from the axioms. If we want the properties given by axioms 1-9 to be true for the real numbers (and we do), then there is no choice other than to have $(-a)(-b) = ab$ and $(a/c)+(b/d) = (ad+bc)/cd$ (see the following theorem). We won't emphasise the idea of making deductions from the axioms. Nonetheless, you should have some appreciation of the ideas involved, and thus you should work through a couple of proofs.

Theorem 1.1. *If a, b, c, d are real numbers and $c \neq 0, d \neq 0$ then*

1. $ac = bc$ *implies* $a = b$.
2. $a0 = 0$
3. $-(-a) = a$
4. $\left(c^{-1}\right)^{-1} = c$
5. $(-1)a = -a$

6. $a(-b) = -(ab) = (-a)b$
7. $(-a) + (-b) = -(a + b)$
8. $(-a)(-b) = ab$
9. $(a/c)(b/d) = (ab)/(cd)$
10. $(a/c) + (b/d) = (ad + bc)/cd$

1.2 $\mathbb{R}$ is a totally ordered field

Order consequences

All the standard properties of inequalities for the real numbers follow from axioms 1-13.
More definitions:
One defines " $>$ ", " $\leq$ " and " $\geq$ " in terms of $<$ as

$$
\begin{aligned}
&a > b \text{ if } b < a,\\
&a \leq b \text{ if } (a < b \text{ or } a = b),\\
&a \geq b \text{ if } (a > b \text{ or } a = b).
\end{aligned}
$$

(Note that the statement $1 \leq 2$, although it isn't one we are likely to make, is indeed true. Why?)
We define $\sqrt{b}$, for $b \geq 0$, to be the number $c \geq 0$ such that $c^2 = b$. Similarly, if n is a natural number, then $\sqrt[n]{b}$ is the number $c \geq 0$ such that $c^n = b$. To prove such a number c always exists requires the "completeness axiom" (see later). To prove the uniqueness of such a number requires the "order axioms". If $0 < a$, we say a is positive and if $a < 0$, we say a is negative.

Some properties of inequalities:
The following are consequences of the axioms which are provided without proofs.

Theorem 1.2. *If a, b and c are real numbers then*

1. $a < b$ *and* $c < 0$ *implies* $ac > bc$
2. $0 < 1$ *and* $-1 < 0$
3. $a > 0$ *implies* $1/a > 0$
4. $0 < a < b$ *implies* $0 < 1/b < 1/a$

1.2.1 Order relation

We defined in $\mathbb{R}$ an order relation $\leqslant$ by $a \leqslant b$ or $b \geqslant a$. We recall the axioms:

$$(A_1) : \forall a \in \mathbb{R} : a \leqslant a \ \ (Reflexive) .$$

$$(A_2) : a \leqslant b \ and \ a \leqslant b \Longleftrightarrow a = b \, (Antisymmetric) .$$

$$(A_3) : a \leqslant b \ and \ b \leqslant c \Longrightarrow a \leqslant c \, (transitive) .$$

We can show that all elements of $\mathbb{R}$ are comparable with respect to the order relation $\leqslant$ and as such we imply a totally ordered set (total order relation)

1.2.2 Example

One can show the utility of such order relation
1. Let a be a real number such that $|a| < \varepsilon, \forall \varepsilon > 0$, we have $a = 0$ (elsewhere, if we choose $\varepsilon = \dfrac{|a|}{2}$, contradiction.
2. Let a and b be two real numbers such that $a < b + \varepsilon, \forall \varepsilon > 0$, then $a \leqslant b$ (else, if we take $\varepsilon = \dfrac{b-a}{2}$, we get a contradiction).

1.3 Natural Numbers and Induction

Definition 1.1. *Mathematical induction is a mathematical proof technique requiring essentially that a statement $P(n)$ holds for every natural number $n = 0, 1, 2, 3, ...$; that is, the overall statement is a sequence of infinitely many cases*
$P(0), P(1), P(2), P(3), ...$

A proof by induction consists of two steps: first step, the base, proves the statement for $n = 0$ without assuming any knowledge of other cases. Second step, the induction, proves that if a statement holds for any given case $n = k$, then it must also hold for the next case $n = k+1$. These two steps establish that the statement holds for every natural number n. The base case doesn't necessarily begin with $n = 0$, but often with $n = 1$ and possibly with any fixed natural number $n = N$, establishing the truth of statement for all natural numbers $n \geqslant N$.

Example 1.1. *Prove the following statements:*

a- $1 + 3 + 5 + ... + (2n - 1) = n^2$

b- $1^2 + 2^2 + 3^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}$

c- $1^3 + 2^3 + 3^3 + ... + n^3 = \left(\frac{n(n+1)}{2}\right)^2$

Example 1.2. *Prove that for all numbers $x$different from* 1*:* $(1+x)(1+x^2)(1+x^4)...(1+x^2n) = \frac{1-x^2n+1}{1-x}$

1.4 Absolute value

The absolute value (or modulus) for any real number a, denoted by $|a|$, is defined as

$$a \longrightarrow |a| \in \mathbb{R}^+ = \begin{cases} a, & \text{si } a \geqslant 0 \\ -a, & \text{si } a < 0 \end{cases},$$

$$|a| = \sup(a, -a).$$

The absolute value has the following fundamental properties:

$$(1) \quad |a| = 0 \Longleftrightarrow a = 0.$$

$$(2) \quad |a| = |-a|.$$

$$(3) \quad |a.b| = |a|.|b|.$$

$$(4) \text{ si } a > 0 : |x| \leqslant a \Longleftrightarrow -a \leqslant x \leqslant a.$$

Indeed, if $x \geqslant 0$ we have $x \geqslant -a$ and

$$|x| \leqslant a \Longleftrightarrow x \leqslant a.$$

if $x \leqslant 0$ we have $x \leqslant a$ and

$$|x| \leqslant a \Longleftrightarrow -x \leqslant a \text{ soit } x \geqslant -a.$$

$$(5) \quad |a+b| \leqslant |a| + |b|.$$

Indeed, if a and b have the same sign, then the inequality is true. If $a \leqslant 0 \leqslant b$, then $a+b \leqslant b \leqslant b+|a|$ (because $a \leqslant 0 \leqslant |a|$), $|a| = -a$. i.e. $a+b \leqslant |a|+|b|$. also $b \geqslant 0 \geqslant -|b|$, $a+b \geqslant a \geqslant a-|b|$. which means that $a+b \geqslant -|a|-|b|$, and using (4), we get

$$|a+b| \leqslant |a| + |b|.$$

We can show, by induction that

$$|a_1 + a_2 + ... + a_n| \leqslant |a_1| + |a_2| + ... + |a_n|.$$

(6) $||a| - |b|| \leqslant |a - b|$.

$|a| = |b + (a - b)|$ and $|b| = |a + (b - a)|$, then, by (5), we obtain

$$|a| \leqslant |b| + |a - b|$$

$$|b| \leqslant |a| + |b - a|$$

$$-|a - b| \leqslant |a| - |b| \leqslant |a - b|,$$

by (4), we claim

$$||a| - |b|| \leqslant |a - b|.$$

1.5 Intervals

In mathematics, a (real) interval is a set that contains all real numbers lying between any two numbers. more precisely

Definition 1.2. *Let a and b bet two real numbers such that $b > a$. The set $\{x : a < x < b\}$ is called open interval and it noted by $]a, b[$. The set $[a, b] = \{x : a \leqslant x \leqslant b\}$ is called closed interval (compact interval). The sets $[a, b[= \{x : a \leqslant x < b\}$, $]a, b] = \{x : a < x \leqslant b\}$, are called (respectively right and left) half-open intervals.*

For all intervals, the points a and b are called endpoints. If $a = b$, we put by definition $[a, a] = \{a\}$ (*degenerate closed interval*) and $]a, a[= \phi$. The length of the interval (closed, open, or half-open) is given by the real number $b - a$.
Examples: 1- The set $\{x : x \leqslant a\}$ is a left unbounded closed interval, noted $]-\infty, a]$. 2- The set $\{x : x < a\}$ is a left unbounded open interval, noted $]-\infty, a[$. 3- The set $\{x : x \geqslant a\}$ is a right unbounded closed interval, noted $[a, +\infty[$. 4- The set $\{x : x > a\}$ is a right unbounded open interval, noted $]a, +\infty[$. 5- The set $\mathbb{R}$ is also noted $]-\infty, +\infty[$. $-\infty$ and $+\infty$ represent infinity numbers.

1.6 Archimedean property

This property does not follow from the algebraic and order axioms alone. It states, informally, that there are no real numbers beyond all the natural numbers.

1.6.1 Archimedean axiom

For every real number a there is a natural number n such that $a < n$. Equivalently, the set $\mathbb{N}$ is not bounded above. We say that $\mathbb{R}$ is Archimidean.

Corollary 1.1. *For all real numbers a and b such that $a > 0$, there exists $n \in \mathbb{N}$ such that $na > b$.*

Proof. Just replace in the axiom a by $\frac{b}{a}$. $\square$

Remark 1.1. *This property seems trivial, actually it's very important, and it allows us to define the famous definition of the integer part of a real number.*

Proposition 1.1. *Let $x \in \mathbb{R}$, there exists a unique integer (called integer part) denoted by $E(x)$, or $[x]$, such that:*

$$E(x) \leqslant x \leqslant E(x) + 1.$$

Example 1.3. $E(e) = 2,\ E(-e) = -3,\ E(1,45632) = 1.$

1.7 The completeness Axiom

In this section we give the completeness Axiom for $\mathbb{R}$. This Axiom will guarantee that $\mathbb{R}$ has no "gaps".

Definition 1.3. *Let S be a nonempty subset of $\mathbb{R}$.*

a. *If S contains the largest element s_0 [that is, s_0 belongs to S and $s \leqslant s_0$ for all $s \in S$, then we call s_0 the maximum of S and write $s_0 = \max S$.*

b. *If S contains the smallest element then we call the smallest element the minimum of S and write $\min S$.*

Example 1.4. *The set $\mathbb{R}$ has no maximum and minimum.*

Example 1.5. *The interval $]a, b[$ has no maximum nor minimum.*

Example 1.6. $\mathbb{N} = \{0, 1, ...\}$, $\min \mathbb{N} = 0$, $\mathbb{N}$ *has no maximum.*

Example 1.7. $\min [a, b[= a$, $\max [a, b[$ *does not exist.*

Example 1.8. *Let A be a subset of $\subset \mathbb{R}$, defined by $A = \{x \in \mathbb{R} : 0 \leqslant \ln x < 1\}$ Check the min and the max for A. Indeed, one has $0 \leqslant \ln x < 1$, which is equivalent to*

$$1 \leqslant x < e.$$

So $A = [1, e[$, we get, $\min A = 1$ *and* $\max A$ *does not exist.*

Definition 1.4. *Let S be a nonempty subset of* $\mathbb{R}$.

a. *If a real number M satisfies $s \leqslant M$ for all $s \in S$, then M is called an upper bound of S and the set S is said to be bounded above.*

b. *If a real number m satisfies $m \leqslant s$ for all $s \in S$, then m is called a lower bound of S and the set S is said to be bounded below.*

c. *The set S is said to be bounded if it is bounded above and bellow. Thus S is bounded if there exist real numbers m and M such that $S \in [m, M]$.*

Example 1.9. *The set $A = \{\sin n,\ n \in \mathbb{N}\}$ is bounded, because*

$$\forall n \in \mathbb{N} : -1 \leqslant \sin n \leqslant 1.$$

Example 1.10. *The set $A = [1, 3]$ is bounded, we can easily check that*

$$\forall x \in A : 1 \leqslant x \leqslant 3.$$

Example 1.11. *Let $A = \left\{\frac{1}{n+1},\ n \geqslant 1\right\}$. A given real $M \geqslant \frac{1}{2}$ is an upper bound for A, and a given real $m \leqslant 0$ is a lower bound for A.*

Definition 1.5. *Let S be a nonempty subset of* $\mathbb{R}$

a. *If S is bounded above and S has a least upper bound, then we will call it the supremum of S and denote it by* $\sup S$.

b. If S is bounded below and S has a greatest lower bound, then we will call it the infimum of S and denote it by $\inf S$.

Example 1.12. *Let $A =]1, 2[$, we have*

$$\forall x \in A : 1 \leqslant x \leqslant 2,$$

hence $\inf A = 1$, $\sup A = 2$.

Example 1.13. *For the set $A = \left\{\frac{n-1}{n},\ n \geqslant 1\right\}$, one has* $\inf A = 0$, $\sup A = 1$ *because*

$$\forall n \geqslant 1 : 0 \leqslant \frac{n-1}{n} \leqslant 1.$$

Example 1.14. *Let $A = \left\{\dfrac{1}{n}, n = 1, 2, 3, 4\right\}$, we have* $\inf A = \min A = \frac{1}{4}$ *and* $\max A = \sup A = 1$.

Theorem 1.3. *Every nonempty subset S of $\mathbb{R}$ that is bounded above has a least upper bound. In other words, $\sup S$ exists and is a real number.*

Corollary 1.2. *Every nonempty subset S of $\mathbb{R}$ that is bounded below has a greatest lower bound* $\inf S$.

Exercise 1.1. *Let A and B be nonempty subsets of real numbers such that $A \subset B$, prove that*

1. *If B is upper bounded, the $\sup B$ exists and $\sup A \leq \sup B$.*

2. *If B is lower bounded, the $\inf B$ exists and $\inf B \leq \inf A$.*

Exercise 1.2. *Let A and B be nonempty subsets of real numbers, show that*

1. $\sup(A \cup B) = \max\{\sup A, \sup B\}$.

2. $\inf(A \cup B) = \min\{\inf A, \inf B\}$.

3. *If $A \cap B \neq= \emptyset$, show that $\sup(A \cap B) \leq \min\{\sup A, \sup B\}$ and $\inf(A \cap B) \geq \max\{\inf A, \inf B\}$.*

1.8 Characterization of supremum and infimum

Theorem 1.4. *Let X be a subset of $\mathbb{R}$. The real M is the supremum for X, if and only if the following hold :*

a. $\forall x \in X$, *x satisfies* $x \leqslant M$.

b. $\forall \varepsilon > 0, \exists\ x_\varepsilon \in X$, *satisfying* $M - \varepsilon < x_\varepsilon \leq M$.

Remark 1.2. *(a) means that M is an upper bound for X. (b) indicates that $M - \varepsilon$ is not an upper bound for all $\varepsilon > 0$.*

Example 1.15. *Let be* $A = \left\{1 - \frac{1}{n}, n \geqslant 1\right\}$. *Prove that* $\sup A = 1$. *Using the upper bound characterization, one can see that 1)* $\forall x = 1 - \frac{1}{n} \in A : x = 1 - \frac{1}{n} \leqslant 1$. *2) We claim that :* $\forall \varepsilon > 0,\ \exists x_\varepsilon \in A : x_\varepsilon > 1 - \varepsilon$

$$\begin{aligned} x_\varepsilon &> 1 - \varepsilon \Leftrightarrow 1 - \frac{1}{n} > 1 - \varepsilon \\ &\Leftrightarrow \frac{1}{n} < \varepsilon \\ &\Leftrightarrow n > \frac{1}{\varepsilon}. \end{aligned}$$

So, let $\varepsilon > 0$ and $n \in N$ such that $n > \frac{1}{\varepsilon}$ (n exists because $\mathbb{R}$ is Archimedean set, then

$$x_\varepsilon = 1 - \frac{1}{n} > 1 - \varepsilon.$$

Thus $\forall \varepsilon > 0,\ \exists x_\varepsilon \in X : x_\varepsilon > 1 - \varepsilon$ *which means* $\sup A = 1$.

Theorem 1.5. *Let X be a subset of $\mathbb{R}$. The real m is the infimum for X, if and only if the following hold :*

a. $\forall x \in X$, *x satisfies* $x \geqslant m$.

b. $\forall \varepsilon > 0, \exists\ x_\varepsilon \in X$, *satisfying* $x_\varepsilon < m + \varepsilon$.

Example 1.16. *Let be* $A = \left\{1 + \frac{1}{n}, n \geqslant 1\right\}$. *Prove that* $\inf A = 1$.
We have 1) $\forall x \in A : x = 1 + \frac{1}{n} \geqslant 1$. *2) We prove that* $\forall \varepsilon > 0,\ \exists x_\varepsilon \in A : m \leq x_\varepsilon < 1 + \varepsilon.$,

indeed

$$\begin{aligned}
x_\varepsilon &< 1+\varepsilon \Leftrightarrow 1+\frac{1}{n} < 1+\varepsilon \\
&\Leftrightarrow \frac{1}{n} < \varepsilon \\
&\Leftrightarrow n > \frac{1}{\varepsilon}.
\end{aligned}$$

So, let be $\varepsilon > 0$ and $n \in N$ such that $n > \frac{1}{\varepsilon}$. Then $x_\varepsilon < 1+\varepsilon$. Thus $\inf A = 1$.

Theorem 1.6. *Let A, B be two nonempty subsets of $\mathbb{R}$. Define*

$$A+B := \{x+y : x \in A \;\; \textit{and} \;\; y \in B\},$$

and

$$A-B := \{x-y : x \in A \;\; \textit{and} \;\; y \in B\}.$$

we have

$$\sup(A+B) = \sup A + \sup B \qquad \textit{and} \qquad \sup(A-B) = \sup A - \inf B.$$

Establish similar formulas for $\inf(A+B)$ and $\inf(A-B)$.

1.9 Extended real number line

Definition 1.6. *The extended real number line is obtained from the real number line $\mathbb{R}$ by adding two infinity elements $+\infty$ and $-\infty$ endowed by the totally order relation extended from that of $\mathbb{R}$ to $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$, where $\overline{\mathbb{R}}$ denotes the extended real number line.*

Operations on $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ are defined by

$$\begin{aligned}
x + (+\infty) &= +\infty + x = +\infty, \forall x \in \mathbb{R}, \\
x + (-\infty) &= -\infty + x = -\infty, \forall x \in \mathbb{R},
\end{aligned}$$

$$x\,(\pm\infty) = (\pm\infty)\,x = \begin{cases} \pm\infty \text{ si } x > 0 \\ \mp\infty \text{ si } x < 0 \end{cases},$$

$$
\begin{aligned}
(+\infty)+(+\infty) &= +\infty, \\
(-\infty)+(-\infty) &= -\infty, \\
(\pm\infty)(\pm\infty) &= +\infty, \\
(\pm\infty)(\mp\infty) &= -\infty.
\end{aligned}
$$

As the sum $(+\infty)+(-\infty)$ and the product $0\,(\pm\infty)=+\infty$ are not well defined, so $\overline{\mathbb{R}}$ does not have any algebraic structures.

1.10 Topology of the line $\mathbb{R}$

1.10.1 Open sets, closed sets, neighbourhood

Definition 1.7. *A subset A of $\mathbb{R}$ is said to be open if it is empty or if for every $x \in A$ there exists an open interval containing x and contained in A.*

In other words an open set in $\mathbb{R}$ is a set which is the union of open intervals. The following assertions are an almost immediate consequence of this definition.

$\mathbf{O}_1$ Every union (finite or infinite) of open sets is open;

$\mathbf{O}_2$ Every finite intersection of open sets is open;

$\mathbf{O}_3$ The line $\mathbb{R}$ and the empty set $\emptyset$ are open sets.

Property O_1 results from the fact that every union of sets, each of which is a union of open intervals , is itself a union of open intervals. To prove the property O_2, it is sufficient to prove it for the intersection of two open sets A, B : By hypothesis

$$A=\cup_i A_i, B=\cup_j B_j$$

where A_i and B_j are open intervals. Therefore

$$A\cap B=(\cup_i A_i)\cap(\cup_j B_j)=\cup_{i,j}(A_i\cap B_j).$$

Since each of the sets $A_i \cap B_j$ is either empty or an open interval, $A\cap B$ is open. Finally, property O_3 is obvious.

Example 1.17. *Every open interval is an open set.*

Example 1.18. *The union of the open interval $]n, n+1[$ where $n \in \mathbb{Z}$, is an open set.*

The intersection of infinite number of open sets is not always open. For example $\cap_{n\in\mathbb{N}^*}\left(\left]\frac{-1}{n},\frac{1}{n}\right[\right)=\{0\}$.

Definition 1.8. *A subset A of $\mathbb{R}$ is said to be closed when its complement $C_{\mathbb{R}}^{A}$ is open.*

Each of the properties O_1, O_2, O_3 at one implies a dual property for closed sets.

Example 1.19. *Every closed interval $[a, b]$ (where $a \leq b$) is a closed set. Indeed, the complement of $[a, b]$ is the union of the two open intervals $]-\infty, a[$ and $]b, +\infty[$, and is therefore an open set.*

It should be observed that a set can be neither open nor closed.

Neighbourhood (or neighborhood): Let $x \in \mathbb{R}$ and $\varepsilon > 0$. A neighbourhood of x is a subset of $\mathbb{R}$ which contains an open interval $V_{(x,\varepsilon)} =]x - \varepsilon, x + \varepsilon[$, containing x.

Example 1.20. *The interval $]-\varepsilon, \varepsilon[$ $(\varepsilon > 0)$ is a neighbourhood of 0. The interval $]-\frac{1}{n}, \frac{1}{n}[$ $(n > 0)$ is a neighbourhood of 0.*

Properties

1) The intersection of finite neighbourhoods of a point x is also its neighbourhood

2) If x and y are two distinct real numbers of $\mathbb{R}$, there exist two neighbourhoods V of x and W of y such that $V \cap W = \emptyset$. ($\mathbb{R}$ is a separated space (or Hausdorff).

Proposition 1.2. *A subset S is open if and only if S a neighbourhood of all the points of S.*

Example 1.21. *If $a, b \in \mathbb{R}$ $(a < b)$, the intervals $]a, b[$, $]-\infty, a[$, $]a, +\infty[$ are neighbourhoods for all their points.*

Remark 1.3. *$\mathbb{R}$ is both open and closed subset, elsewhere $[a, b[$ $(a < b)$ is neither open nor closed.*

The meaning which we have just given to the word "neighbourhood" appears different from the one defined in ordinary usage, since for us a point x of $\mathbb{R}$ has many neighbourhoods, and one of them is the space $\mathbb{R}$ itself.

Accumulation points of a set

Definition 1.9. *If A is a subset of $\mathbb{R}$, a point x of $\mathbb{R}$ is called an accumulation point of A if, in every neighbourhood of x, there exists at least one point of A different from x. In other words, if A is a subset of $\mathbb{R}$, a point x of $\mathbb{R}$ is called an accumulation point of A if, $\forall \varepsilon > 0$, $A \cap]x - \varepsilon, x + \varepsilon[/ \{x\} \neq \emptyset$.*

The set of accumulation points is denoted by A' (it can be empty).

Example 1.22. $A = [1, 2]$, *then* $A' = [1, 2]$.

Example 1.23. $A =]0, 1[\cup \{2\}$, *so* $A' = [0, 1]$.

Example 1.24. $A = \{1, 2, 3, 4\}$, *therefore* $A' = \phi$.

Example 1.25. $A = \left\{\frac{1}{n}, n \geqslant 1\right\}$, *thus* $A' = \{0\}$.

Remark 1.4. *An accumulation point of a set does not necessarily belong to the set. For example, the point 0 is an accumulation point of the set of point* $A = \left\{\frac{1}{n}, n \geqslant 1\right\}$, *but does not belong to this set. Again 0 and 1 are accumulation points of* $]0, 1[$ *without belonging to this interval.*

Proposition 1.3. *Every closed set contains its accumulation points. Conversely, every set which contains its accumulation points is closed.*

Proof. Let A be a closed set; if $x \in C_{\mathbb{R}}^{A}$, then the open set $C_{\mathbb{R}}^{A}$ is a neighbourhood of x and does not contain any point of A nor does it contain any point of A. Thus x cannot be an accumulation point of A. Conversely, if A is such that no point of $C_{\mathbb{R}}^{A}$ are an accumulation point of A, then there exists for each $x \in C_{\mathbb{R}}^{A}$ a neighbourhood of x not containing any point of A, and therefore contained in $C_{\mathbb{R}}^{A}$; the set $C_{\mathbb{R}}^{A}$ is thus a neighbourhood of each of its points, i.e., it is open; in other words, A is closed. $\square$

Isolated points

Definition 1.10. *An isolated point of a set* A *is a point* x *of* A *which is not an accumulation point of* A*. In other words, it is a point* x *of* A *which has a neighbourhood* V *such that* $A \cap V = x$*. (*$\exists \varepsilon > 0,\ A \cap]x - \varepsilon, x + \varepsilon[= \{x\}$.

Example 1.26. *Let* $A = [0, 1] \cup \mathbb{N}$, *then the isolated points of* A *are* $\{2, 3, ..., n, ...\}$.

CHAPTER

2

COMPLEX NUMBERS

2.1 Algebraic properties

Let (x,y) and (x',y') be two elements of $\mathbb{R}^2$. We define two operations on $\mathbb{R}^2$, by setting $(x,y)\times(x',y') = (xx'-yy', xy'+yx')$ et $(x,y)+(x',y') = (x+x', y'+y')$. This two composition operations define a new field, which is the complex commutative field denoted by $\mathbb{C}$. The additive neutral element is given by $0=(0,0)$ and the multiplicative neutral element is $(1,0)$, the multiplicative inverse of $(x,y)\neq(0,0)$ is $\left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right)$. By identifying $(x,0)\in\mathbb{R}^2$ with $x\in\mathbb{R}$, and by setting $i=(0,1)$.

$$\mathbb{C} = \left\{z/\ z = x+iy \text{ with } x,y\in\mathbb{R} \text{ and } i^2=-1\right\}.$$

So we do calculus with complex numbers as what we do with real numbers taking into account that $i^2=-1$.

Example 2.1. *For all $n\in\mathbb{N}$, we have*

$$1+i+i^2+\ldots+i^n = \frac{1-i^{n+1}}{1-i} = \begin{cases} 1 & \textit{if } n=4k \\ 1+i & \textit{if } n=4k+1 \\ i & \textit{if } n=4k+2 \\ 0 & \textit{if } n=4k+3 \end{cases}$$

Definition 2.1. *A complex number is any number of the form $z = a + ib$ where a and b are real numbers and i is the imaginary unit. The notations $a + ib$ and $a + bi$ are used interchangeably. The real number a in $z = a + ib$ is called the real part of $z = a + ib$; the real number b is called the imaginary part of $z = a + ib$. The real and imaginary parts of a complex number $z = a + ib$ are abbreviated $Re\,(z)$, and $Im\,(z)$, respectively.*

Definition 2.2. *Complex numbers $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ are equal, $z_1 = z_2$, if $a_1 = a_2$ and $b_1 = b_2$.*

Conjugate and modulus of a complex number

Definition 2.3. *Let $z = x+iy$ be a complex number, we define the conjugate of $\overline{z} = x+iy$ by $\overline{z} = x - iy$. The positive number $|z| = \sqrt{z\overline{z}} = \sqrt{x^2 + y^2}$ denotes its modulus.*

The modulus $|z|$ of a complex number z is also called the absolute value of z. We shall use both words modulus and absolute value throughout this text.

Example 2.2. *If $z = 2 - 3i$, then we find the modulus of the number to be $|z| = \sqrt{2^2 + (-3)^2} = \sqrt{13}$. If $z = -9i$, then $|z| = |-9i| = \sqrt{(-9)^2} = 9$.*

The following properties hold

1) $\overline{z_1 \pm z_2} = \overline{z_1} \pm \overline{z_2}$,

2) $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$,

3) $\overline{\dfrac{z_1}{z_2}} = \dfrac{\overline{z_1}}{\overline{z_2}}$.

4) $|z| \geqslant 0$ and $|z| = 0 \Leftrightarrow z = 0$,

5) $|z_1 z_2| = |z_1|\,|z_2|$

6) $||z_1| - |z_2|| \leqslant |z_1 - z_2| \leqslant |z_1| + |z_2|$

7) $|Rez| \leqslant |z|\,,\ |Imz| \leqslant |z|\,.$

8) $\dfrac{1}{z} = \dfrac{\overline{z}}{|z|}$.

2.2 Complex plane

A complex number $z = x + iy$ is uniquely determined by an ordered pair of real numbers (x, y). The first and second entries of the ordered pairs correspond, in turn, with the real and imaginary parts of the complex number. For example, the ordered pair $(2, -3)$ corresponds to the complex number $z = 2 - 3i$. Conversely, $z = 2 - 3i$ determines the ordered pair $(2, -3)$. The numbers $7, i$, and $-5i$ are equivalent to $(7, 0), (0, 1), (0, -5)$, respectively. In this manner we are able to associate a complex number $z = x + iy$ with a point (x, y) in a coordinate plane.

Complex plane

Because of the correspondence between a complex number $z = x + iy$ and one and only one point (x, y) in a coordinate plane, we shall use the terms complex number and point interchangeably. The coordinate plane is called the complex plane or simply the z -plane. The horizontal or $x-axis$ is called the real axis because each point on that axis represents a real number. The vertical or $y - axis$ is called the imaginary axis because a point on that axis represents a pure imaginary number.

Vector

A complex number $z = x + iy$ can also be viewed as a two dimensional position vector, that is, a vector whose initial point is the origin and whose terminal point is the point (x, y). This vector interpretation prompts us to define the length of the vector z as the distance $|z| = \sqrt{z\bar{z}} = \sqrt{x^2 + y^2}$ from the origin to the point (x, y) . This length is the modulus.

2.3 Polar Form of Complex Numbers

Recall that a point P in the plane whose rectangular coordinates are (x, y) can also be described in terms of polar coordinates. The polar coordinate system, invented by Isaac Newton, consists of point O called the pole and the horizontal half-line emanating from the pole called the polar axis. If r is a directed distance from the pole to P and θ is an angle of inclination (in radians) measured from the polar axis to the line OP, then the point can be described by the ordered pair (r, θ), called the polar coordinates of P.

Polar form

Suppose, that a polar coordinate system is superimposed on the complex plane with the polar axis coinciding with the positive x-axis and the pole O at the origin. Then x, y, r and θ are related by $x = rcos\theta, y = rsin\theta$. These equations enable us to express a nonzero complex number $z = x+iy$ as $z = (rcos\theta)+i(rsin\theta)$. We say that $z = r(cos\theta+isin\theta)$ is the polar form or polar representation of the complex number z. Again, the coordinate r can be interpreted as the distance from the origin to the point (x, y) . In other words, we shall adopt the convention that r is never negative so that we can take r to be the modulus of z, that is, $r = |z|$. The angle θ of inclination of the vector z, which will always be measured in radians from the positive real axis, is positive when measured counterclockwise and negative when measured clockwise. The angle θ is called an argument of z and is denoted by $\theta = arg(z)$. An argument θ of a complex number must satisfy the equations $cos\theta = \frac{x}{r}$ and $sin\theta = \frac{y}{r}$. An argument of a complex number z is not unique since $cos\theta$ and sin $sin\theta$ are $2\pi - periodic$. In practice we use $tan\theta = \frac{y}{x}$ to find θ. However, because $tan\theta$

is $\pi - periodic$, some care must be exercised in using the last equation. The following example illustrates how this is done.

Example 2.3. *Express $-\sqrt{3} - i$ in polar form.*

Solution

With $x = -\sqrt{3}$ and $y = -1$ we obtain $r = |z| = 2$. Now $\frac{y}{x} = \frac{-1}{-\sqrt{3}} = \frac{1}{\sqrt{3}}$, and so a $\left(tan\frac{1}{\sqrt{3}}\right)^{-1} = \pi/6$, which is an angle whose terminal side is in the first quadrant. But since the point $(-\sqrt{3}, -1)$ lies in the third quadrant, we take the solution of $tan\frac{1}{\sqrt{3}}$ to be $\theta = arg(z) = \frac{\pi}{6} + \pi = \frac{7\pi}{6}$. It follows that a polar form of the number is $z = 2(cos\frac{7\pi}{6} + isin\frac{7\pi}{6})$.

Principal Argument

The symbol $arg(z)$ actually represents a set of values, but the argument θ of a complex number that lies in the interval $-\pi < \theta < \pi$ is called the principal value of $arg(z)$ or the principal argument of z. The principal argument of z is unique and is represented by the symbol $Arg(z)$, that is, $-\pi < Arg(z) \leqslant \pi$. For example, if $z = i$, we have some values of $arg(i)$ as $\frac{\pi}{2}, \frac{5\pi}{2}, \frac{-3\pi}{2}$, and so on, but $Arg(i) = \frac{\pi}{2}$. Similarly, we can verify that the principal argument of $-\sqrt{3} - i$ is $Arg(z) = \frac{\pi}{6} - \pi = \frac{-5\pi}{6}$. Using $Arg(z)$ we can express the complex number $-\sqrt{3}-i$ in the alternative polar form $z = 2\left(cos\frac{-5\pi}{6} + isin\frac{-5\pi}{6}\right)$.

Moivre's formula

If $z \neq 0$ we have $z = r(\cos\theta + i\sin\theta)$ where θ is the principal argument. By definition, $-\pi < Arg(z) \leq \pi$. The polar form of a complex number is especially convenient when multiplying or dividing two complex numbers. We can verify that

$$z_1z_2 = r_1r_2\left(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\right).$$

Thus $|z_1z_2| = |z_1||z_2|$ and $\arg(z_1z_2) = (\arg(z_1) + \arg(z_2))\, mod 2\pi$,
and

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\left(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right).$$

continuing in this manner, we obtain a formula for the-nth power of z.

$$(r\cos\theta + ir\sin\theta)^n = r^n\left(\cos(n\theta) + i\sin(n\theta)\right).$$

Euler's formula

Let $z \neq 0$, and $z = r(\cos\varphi + i\sin\varphi)$, if $r = 1$, then $z = \cos\varphi + i\sin\varphi$. Put

$$e^{i\theta} = \cos\theta + i\sin\theta$$

which is called Euler's formula.

Example 2.4. $e^{2\pi i} = 1$, $e^{\pi i} = -1$, $e^{-\frac{\pi}{2}i} = -i$, $e^{\frac{\pi}{2}i} = i$.
Replacing θ by $(-\theta)$ in $e^{i\theta} = \cos\theta + i\sin\theta$, we obtain

$$e^{-i\theta} = \cos\theta - i\sin\theta$$

From the last two formulas above, we deduce Euler's formulas:

$$\cos\varphi = \frac{e^{i\theta} + e^{-i\theta}}{2}, \ \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2}.$$

Properties

1) $e^{i\theta}.e^{-i\theta'} = e^{i(\theta+\theta')}$
2) $\dfrac{e^{i\theta}}{e^{i\theta'}} = e^{i(\theta-\theta')}$
3) $\left(e^{i\theta}\right)^n = e^{in\theta}$, $n \in \mathbb{N}$.

Definition 2.4. *Let $z \neq 0$. Then $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$ is called the exponential form of z.*

The nth root of a complex number
All nonzero complex number $z = re^{i\theta}$ admits n roots $n-$*iémes* w_k where, $w_k = \sqrt[n]{r}e^{i\frac{\theta+2\pi k}{n}}$ with $k \in \{0, 1, ..., n-1\}$, $n > 1$.

Example 2.5. *Solve the following equation $z^4 = 1 + i$.*
The solutions are the fourth roots of $1+i$, so we have $w_k = \sqrt[4]{2}e^{\frac{i}{4}\left(\frac{\pi}{4}+2\pi k\right)}$, $k = 0, 1, 2, 3$.

CHAPTER

3

SEQUENCES OF REAL NUMBERS

3.0.1 General definitions

Definition 3.1. *A real sequence or a sequence of real numbers is defined as a function from* $\mathbb{N}$*, the set of natural number to* $\mathbb{R}$*, the set of real numbers. In other words* $\mathbb{N}$ *:* $n \in \mathbb{N} \longmapsto u_n \in \mathbb{R}$, $u_n = f(n)$. *It is customary to denote a sequence by a letter such as* u *and to denote its value at* n *as* $(u_n)_{n\in\mathbb{N}}$ *or more clearly* (u_n), $u_n = (u_0, u_1, ...)$.

The real numbers $u_0, u_1, ...$ are called elements or terms of the sequence u_n. The number u_n is called the nth term of rank n of the sequence or general term.

Examples:

$$\left(\frac{1}{n+1}\right)_{n\in\mathbb{N}} = \left(1, \frac{1}{2}, \frac{1}{3}, ...\right),$$

$$\left(2^{(-1)^n}\right)_{n\in\mathbb{N}\cup 0} = \left(2, \frac{1}{2}, 2, \frac{1}{2}, ...\right)$$

Definition 3.2. *Let* $\{u_n\}_{n\in\mathbb{N}}$ *be a real sequence, it is said to be :*

- constant if there exists $a \in \mathbb{R}$ *such that:* $\forall n \in \mathbb{N} : u_n = a$.

- stationary if there exists $n_0 \in \mathbb{N}$ *such that* $\forall n \geqslant n_0 : u_n = u_{n_0}$.

- *truncated, if its general term can only be defined after a certain value of* n, *let* $n \geqslant N_0$. *If, necessary we can complete by putting* $u_n = 0$ *pour* $n < N_0$.

Example 3.1. $\left(sin\left(2n+\frac{1}{2}\right)\pi\right) = (1,1,1,...)$ *is a constant sequence.*

$\left(sinn!\frac{\pi}{5}\right)_{n\in\mathbb{N}} = \left(\sin\frac{\pi}{5}, \sin\frac{\pi}{5}, \sin 2\frac{\pi}{5}, \sin\left(6\frac{\pi}{5}\right), \sin\left(24\frac{\pi}{5}\right), 0, 0, ...\right)$ *is a stationary sequence.*

$\left(\frac{1}{n(n-3)}\right)$ *is a truncated sequence, it is defined after* $n = 4$.

3.0.2 Bounded sequences, convergent sequences

Definition 3.3. *A sequence* (u_n) *is said to be upper bounded (resp. lower bonded), if there exists* $M \in \mathbb{R}$ *such that* $\forall n \in \mathbb{N} : u_n \leqslant M$ *(resp.* $u_n \geqslant M$ *).*
A sequence is said to be bounded if it is upper and lower bounded.

in other terms (u_n) *is said to be bounded if there exists* $M > 0, M \in \mathbb{R}$, *such that* $\forall n \in \mathbb{N} : |u_n| \leqslant M$.

Example 3.2. *The sequence* $\left(\frac{1}{n}\right)_{n\in\mathbb{N}}$ *is bounded, because,* $\forall n \geqslant 1 : 0 < u_n = \frac{1}{n} \leqslant 1$.

The sequence $((-1)^n)_{n\in\mathbb{N}}$ *is bounded because* $\forall n \in \mathbb{N} : |(-1)^n| = 1$.

The sequence $(n)_{n\in\mathbb{N}}$ *is lower bounded, but the sequence* $((-1)^n n)$ *is neither upper bounded nor lower bounded.*

Definition 3.4. *We say that the sequence* (u_n) *converges to the limit* l *as* n *approaches infinity, and write* $\lim_{n\to+\infty} u_n = l$ *or* $u_n \to l$, $n \to +\infty$, *if*

$$\forall \varepsilon > 0, \exists N_0 = N_0(\varepsilon) \in \mathbb{N}, \forall n > N_0(\varepsilon) \Rightarrow |u_n - l| < \varepsilon. \tag{3.1}$$

A sequence that does not converge to some real number is said to diverge.
Resume

$$\lim_{n\to+\infty} u_n = l \Leftrightarrow \forall \varepsilon > 0, \exists N_0(\varepsilon) \in \mathbb{N}, \forall n > N_0(\varepsilon) \Rightarrow |u_n - l| < \varepsilon.$$

Theorem 3.1. *The limit of a convergent sequence is unique.*

Proof. By contradiction technique, we suppose that we have two limits l_1, l_2, we must show that $l_1 = l_2$, indeed by the definition of limit there must exist N_1 so that

$$\forall n > N_1(\varepsilon) \Rightarrow |u_n - l_1| < \frac{\varepsilon}{2},$$

and must exist N_2 so that

$$\forall n > N_2(\varepsilon) \Rightarrow |u_n - l_2| < \frac{\varepsilon}{2}.$$

For $n > max\{N_1, N_2\}$, the triangle shows that

$$|l_1 - l_2| = |l_1 - l_2 + u_n - u_n| < |u_n - l_1| + |u_n - l_2| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon$$

This shows that $|l_1 - l_2| < \varepsilon$ for all positive ε. It follows that $|l_1 - l_2| = 0$. □

Example 3.3. *The following sequence* $((-1)^n)_{n\in\mathbb{N}} = (1, -1, 1, -1, ...)$ *does not converge to any real number.*

3.0.3 Convergent sequences properties

Theorem 3.2. *Convergent sequences are bounded.*

Proof. Let (u_n) be a convergent sequence, and let $\lim\limits_{n\to+\infty} u_n = l$. Applying Definition 3.4 with $\varepsilon = 1$, we obtain $N \in \mathbb{N}$ so that

$$\forall n > N(\varepsilon) \Rightarrow |u_n - l| < 1,$$

From the triangle inequality we see that $N \in \mathbb{N}$ implies

$$\forall n > N \Rightarrow |u_n| < 1 + |l|,$$

define $M = max\{1 + |l|, |u_1|, |u_2|, |u_3|, ..., |u_N|\}$. Then we have $|u_n| \leqslant M$, for all $n \in \mathbb{N}$, so (u_n) is a bounded sequence. □

Remark 3.1. *The boundness of a sequence of real numbers is necessary for convergence but not sufficient.*

Example 3.4. *La suite* $((-1)^n n)_{n\in\mathbb{N}} = (1, -1, 1, -1, ...)$ *is unbounded, so divergent.*

Theorem 3.3. *If* $\lim\limits_{n\to+\infty} u_n = a$, $\lim\limits_{n\to+\infty} v_n = b$, *and* $u_n \leqslant v_n$, $(\forall n \in \mathbb{N})$. *Then* $a \leqslant b$.

Theorem 3.4. *Let be* $\{u_n\}_{n\in\mathbb{N}}$, $\{v_n\}_{n\in\mathbb{N}}$, *two sequences which converge to a and (resp. b), despite that* $(\forall n \in \mathbb{N})$ $(u_n < v_n)$, *we obtain* $a \leqslant b$.

Example 3.5. *Let be the following sequences* $\left(\dfrac{1}{n+1}\right)_{n\in\mathbb{N}}$*, and* $\left(\dfrac{1}{n+2}\right)_{n\in\mathbb{N}}$*,, we have* $\forall n \in \mathbb{N}\ \dfrac{1}{n+2} < \dfrac{1}{n+1}$*, but the two limits are equal to* 0*.*

Theorem 3.5. *(Squeeze Theorem) Suppose that* $\{u_n\}_{n\in\mathbb{N}}$, $\{v_n\}_{n\in\mathbb{N}}$*, and* $\{w_n\}_{n\in\mathbb{N}}$*, are sequences of real numbers such that 1)* $\forall n \in \mathbb{N} : u_n \leqslant w_n \leqslant v_n$*. 2)* $\lim\limits_{n\to+\infty} u_n = \lim\limits_{n\to+\infty} v_n = a$*. Then* $\lim\limits_{n\to+\infty} w_n = a$*.*

Example 3.6. *Find* $\lim\limits_{n\to+\infty} \sum\limits_{k=1}^{n} \dfrac{n}{n^2+k}$*.*

We have $\forall k \in \{1, 2, ..., n\}\ \dfrac{n}{n^2+n} \leqslant \dfrac{n}{n^2+k} \leqslant \dfrac{n}{n^2+1}$

where $\dfrac{n^2}{n^2+n} \leqslant \sum\limits_{k=1}^{n} \dfrac{n}{n^2+k} \leqslant \dfrac{n^2}{n^2+1}$*. Since* $\lim\limits_{n\to+\infty} \dfrac{n^2}{n^2+n} = \lim\limits_{n\to+\infty} \dfrac{n^2}{n^2+1} = 1$*, then by the squeeze technique, we obtain,*
$\lim\limits_{n\to+\infty} \sum\limits_{k=1}^{n} \dfrac{n}{n^2+k} = 1$.

Theorem 3.6. *If* $\lim\limits_{n\to+\infty} u_n = l$*, then* $\lim\limits_{n\to+\infty} |u_n| = |l|$*.*

$$\lim_{n\to+\infty} u_n = l \Rightarrow \lim_{n\to+\infty} |u_n| = |l| .$$

Proof. It follows from the inequality $||x| - |y|| \leqslant |x - y|$. □

Remark 3.2. *The converse is in general wrong.*

Example 3.7. *Consider the sequence defined by* $((-1)^n)_{n\in\mathbb{N}}$*. One has* $\lim\limits_{n\to+\infty} |u_n| = \lim\limits_{n\to+\infty} |(-1)^n| = 1$*, despite that* $\lim\limits_{n\to+\infty} u_n = \lim\limits_{n\to+\infty} (-1)^n$ *does not exist.*

Remark 3.3. $\lim\limits_{n\to+\infty} u_n = 0 \Longleftrightarrow \lim\limits_{n\to+\infty} |u_n| = 0$.

3.0.4 Combination Rules for convergent sequences

Theorem 3.7. *Suppose that the following sequences are convergent* $(u_n)_{n\in\mathbb{N}}$, $(v_n)_{n\in\mathbb{N}}$*. Set* $\lim\limits_{n\to+\infty} u_n = a$, $\lim\limits_{n\to+\infty} v_n = b$*, Then*

1) $\forall c \in \mathbb{R}$, $\lim\limits_{n\to+\infty} cu_n = c \lim\limits_{n\to+\infty} u_n = ca$*.*

2) $\lim\limits_{n\to+\infty} u_n \pm v_n = a \pm b$

3) $\lim\limits_{n\to+\infty} u_n v_n = ab$

4) If $(\forall n \in \mathbb{N})\ (u_n \neq 0)$*, if* $b \neq 0$*, then* $\lim\limits_{n\to+\infty} \dfrac{u_n}{v_n} = \dfrac{a}{b}$*.*

Proof. 1) We assume $c \neq 0$, since this result is trivial for $c = 0$, Let $\varepsilon > 0$ and note that we need to show that $|cu_n - ca| < \varepsilon$ for large n. Since $\lim\limits_{n\to+\infty} u_n = a$, there exists N such that

$$\forall n > N \Rightarrow |u_n - a| < \frac{\varepsilon}{|c|}.$$

Then

$$\forall n > N \Rightarrow |cu_n - ca| < \varepsilon.$$

2) Let $\varepsilon > 0$; we need to show that $|u_n \pm v_n - (a \pm b)| < \varepsilon$, for large n, we note that $|u_n \pm v_n - (a \pm b)| < |u_n - a| + |v_n - b)|$. Since there exists N_1 such that

$$\forall n > N_1 \Rightarrow |u_n - a| < \frac{\varepsilon}{2}.$$

Likewise, there exists N_2 such that

$$\forall n > N_2 \Rightarrow |u_n - a| < \frac{\varepsilon}{2}.$$

Let $N = max\{N_1, N_2\}$. Then clearly

$$\forall n > N \Rightarrow |u_n \pm v_n - (a \pm b)| < \varepsilon.$$

3) The trick here is to look at the inequality

$$|u_n v_n - ab| = |u_n v_n + u_n b - u_n b - ab| < |u_n v_n - u_n b| + |u_n b - ab| = |u_n|\,|v_n - b| + |b|\,|u_n - a|$$

4) To prove (4) it suffices to show that $\lim\limits_{n\to+\infty} \frac{1}{v_n} = \frac{1}{b}$.
The result (4) then follows from (3). Since $b \neq 0$, and $\lim\limits_{n\to+\infty} v_n = b$,, there exists a positive integer N_0 such that $|v_n - b| < \frac{1}{2}|b|$, for all $n \geqslant N_0$.
Also, since $|b| < |v_n - b| + |b| < \frac{1}{2}|b| + |v_n|$, for $n \geqslant N_0$, we have $|v_n| \geqslant \frac{1}{2}|b|$ for all $n \geqslant N_0$. Therefore,
$\left|\frac{1}{v_n} - \frac{1}{b}\right| = \frac{|b-v_n|}{|bv_n|} \leqslant \frac{2}{|b|^2}|v_n - b|$. Let $\varepsilon > 0$ be given. Since $\lim\limits_{n\to+\infty} v_n = b$, we can choose an integer $N1 \geqslant N_0$ so that $|v_n - b| \leqslant \varepsilon \frac{|b|^2}{2}$ for all $n \geqslant N1$.
Therefore
$\left|\frac{1}{v_n} - \frac{1}{b}\right| \leqslant \varepsilon$, for all $n \geqslant N1$. □

Sequences which tend to infinity

$$\lim_{n\to+\infty} u_n = +\infty \Leftrightarrow \forall A > 0, \exists N \in \mathbb{N}, \forall n > N \Rightarrow u_n > A.$$

$$\lim_{n\to+\infty} u_n = -\infty \Leftrightarrow \forall A > 0, \exists N \in \mathbb{N}, \forall n > N \Rightarrow u_n < -A.$$

Example 3.8. *Let $a \in \mathbb{R}$, $a > 1$. Show that $\lim\limits_{n\to+\infty} a^n = +\infty$. Let $A > 0$. Then $a^n > A \Leftrightarrow n > \frac{\ln A}{\ln a}$.*

So $\forall A > 0, \exists N = E\left(\frac{\ln A}{\ln a}\right)$, $\forall n > N = E\left(\frac{\ln A}{\ln a}\right) \Rightarrow a^n > A \Leftrightarrow \lim\limits_{n\to+\infty} a^n = +\infty$ $(a > 1)$.

3.0.5 Monotone sequences

Definition 3.5. *Let* (u_n) *be a sequence of real numbers. We say that* (u_n) *is nondecreasing (resp. nonincreasing) if it satisfies the inequality* $\forall n \in \mathbb{N} : u_n \leqslant u_{n+1}$ *(resp.* $u_n \geqslant u_{n+1}$*).*

If $u_n < u_{n+1}$ *(resp.* $u_n > u_{n+1}$*), we say that* (u_n) *is increasing (resp. decreasing). We say that* (u_n) *is monotone if it is either nonincreasing or nondecreasing. We say that* (u_n) *is strictly monotone if it is either increasing or decreasing.*

Corollary 3.1. *An nondecreasing sequence is lower bounded, and the nonincreasing one is upper bounded.*

Example 3.9. *Let* $\left(\frac{n-1}{n}\right)_{n\in\mathbb{N}}$ *be a real sequence. Since* $u_{n+1} - u_n = \frac{n}{n+1} - \frac{n-1}{n} = \frac{1}{n(n+1)} > 0$, $\forall n \in \mathbb{N}$, *then* $u_{n+1} > u_n$ $(n \in \mathbb{N})$. *So the real sequence* $\left(\frac{n-1}{n}\right)_{n\in\mathbb{N}}$ *is increasing.*

Example 3.10. *The sequence* $\left(\frac{1}{n}\right)_{n\in\mathbb{N}}$ *is decreasing because,* $\forall n \in \mathbb{N} : u_{n+1} = \frac{1}{n+1} < \frac{1}{n} = u_n$.

Remark 3.4. *We check the monotony by evaluating the sign of* $u_{n+1} - u_n$. *If the sequence is positive, we can just compare* $\frac{u_{n+1}}{u_n}$ *with* 1.

Theorem 3.8. *All bounded monotone sequences converge.*

Proof. Let (u_n) be a bounded nondecreasing sequence. Let S denote the set $\{u_n : n \in \mathbb{N}\}$, and let $u = supS$. Since S is bounded, $u = supS$ represents a real number. We show that $limu_n = u$. Let $\varepsilon > 0$. Since $u - \varepsilon$ is not an upper bound for S, there exists N such that $u_N > u - \varepsilon$. Since (u_n) is nondecreasing, we have $u_N \leqslant u_n$ for all $n \geqslant N$. Of course, $u_n \leqslant u$ for all n, $n \geqslant N$ so n > N implies $u - \varepsilon < u_n \leqslant u$, which implies $|u_n - u| < \varepsilon$. This shows that $limu_n = u$. The proof for bounded nonincreasing sequences is left as exercise. □

3.0.6 Adjacent sequences

Definition 3.6. *Let* (u_n) (v_n) *be two real sequences. We say that the two sequences are adjacent if the first is nondecreasing, the second is nonincreasing, and their difference converges to* 0. *In other words*

1) The sequence (u_n) is nondecreasing and the sequence (v_n) is nonincreasing,

2) The difference $(v_n - u_n)$ converges to 0, when n approaches ∞.

Example 3.11. $u_n = 1 + \frac{1}{n^2}$ *and* $v_n = 1 - \frac{1}{n^2}$

Proposition 3.1. *Two adjacent sequences converge, and converge to the same limit.*

Proof. Let (u_n) and (v_n) be two real sequences such as that (u_n) is nondecreasing, the sequence (v_n) is nonincreasing, and $(v_n - u_n)$ converges to 0, when n approaches ∞. We first show that $v_n > u_n$. Put $W_n = v_n - u_n$. We check the sign of $W_{n+1} - W_n = v_{n+1} - u_{n+1} - v_n + u_n = v_{n+1} - v_n - (u_{n+1} - u_n) < 0$, because (u_n) is nondecreasing, and (v_n) is nonincreasing. So (W_n) is nonicreasing and converges to 0, thus (W_n) is positive and $v_n > u_n$. Now we show that (u_n) and (v_n) converge. Indeed, one has $u_n < v_n < v_0$, so u_n is nonincreasing and bounded above by v_0, all bounded monotone sequences converge. Likewise $u_0 < u_n < v_n$, we do and write the same things. v_n converges. The two adjacent sequences admit the same limit. Since $lim u_n = l_1, lim v_n = l_2$, we have $lim W_n = lim(v_n - u_n) = 0 \Leftrightarrow l_1 = l_2$. $\square$

3.0.7 Subsequences

Definition 3.7. *A subsequence of a sequence (u_n) is a sequence formed by deleting elements of the u_n to produce a new u_n. This subsequence is usually written as $v_n = u_{\rho(n)}$, $n \in \mathbb{N}$, where $\rho : \mathbb{N} \to \mathbb{N}$ is an increasing sequence of positive integers.*

Example 3.12. *Let consider the real sequence defined by $((-1)^n)_{n\in\mathbb{N}}$. The sequences defined by :*

$$v_n = u_{2n} = (-1)^{2n} = 1 \, (n \in \mathbb{N})$$

$$w_n = u_{2n+1} = (-1)^{2n+1} = -1 \, (n \in \mathbb{N})$$

are subsequences of the sequence $((-1)^n)_{n\in\mathbb{N}}$.

Remark 3.5. *If $\rho : \mathbb{N} \to \mathbb{N}$ is an increasing application, then $\forall n \in \mathbb{N} : \rho(n) \geqslant n$.*

Corollary 3.2. *Let (u_n) be a real sequence, if u_n converges to l. Then all subsequences $\left(v_n = u_{\rho(n)}\right)$ of (u_n) converges to l.*

Remark 3.6. *The converse is in general wrong.*

Example 3.13. *The divergent sequence $((-1)^n)_{n\in\mathbb{N}}$ admits the following convergent subsequences $u_{2n} = 1 \ (n \in \mathbb{N})$, $u_{2n+1} = -1 \ (n \in \mathbb{N})$.*

Theorem 3.9. *(Bolzano-Weierstrass) Every bounded sequence admits a convergent subsequence.*

Proof. Let $(u_n$ be a real sequence and $m \in \mathbb{N}$. we say that m is a peak of the sequence $(u_n$ if : $n > m \Rightarrow u_n < u_m$. Suppose that (u_n) has an infinite numbers of peaks. $k_0 < k_1 < k_2 < k_3 < k_4 < k_5 < ... < k_n < ...$ and consider the subsequence (u_{k_n}). Then (u_{k_n}) is decreasing since $k_n > k_m \Rightarrow u_{k_n} < u_{k_m}$ and thus (u_{k_n}) is monotone. Suppose that (u_{k_n}) has a finite number of peaks and let N be the last (greatest) peak. Then $k_0 = N+1$ is not a peak and so there exists k_1 such that $u_{k_1} > u_{k_0}$. Having defined k_n such that $k_n > k_{n-1} > N$, then there exists $k_{n+1} > k_n$ such that $u_{k_{n+1}} > u_{k_n}$. The subsequence u_{k_n} is obviously increasing and so it is monotone. Now if u_n is in addition bounded, so is u_{k_n} and applying the Monotone Convergence Theorem yields that the subsequence has a finite limit. □

3.0.8 Cauchy sequences

Definition 3.8. *A sequence (u_n) of real numbers is called a Cauchy sequence if*

$$\forall \varepsilon > 0, \exists N(\varepsilon) \in \mathbb{N}, \forall n, m > N(\varepsilon) \Rightarrow |u_n - u_m| < \varepsilon.$$

Example 3.14. *The real sequence $u_n = \frac{\sin 1}{2} + \frac{\sin 2}{2^2} + ... + \frac{\sin n}{2^n}$, $n \geqslant 1$ is a Cauchy sequence.*

Indeed, for all $(n, m) \in \mathbb{N}^2, n > m$, we have

$$\begin{aligned} |u_n - u_m| &= \left|\frac{\sin(m+1)}{2^{m+1}} + \frac{\sin(m+2)}{2^{m+2}} + ... + \frac{\sin n}{2^n}\right| \\ &\leqslant \frac{1}{2^{m+1}} + \frac{1}{2^{m+2}} + ... + \frac{1}{2^n} \\ &< \frac{1}{2^{m+1}} + \frac{1}{2^{m+2}} + ... + \frac{1}{2^n} + ... \\ &= \frac{1}{2^{m+1}}\left(\frac{1}{1-\frac{1}{2}}\right) = \frac{1}{2^m} \end{aligned}$$

Since $\frac{1}{2^m} \to 0, m \to \infty$, then

$$\forall \varepsilon > 0, \exists N(\varepsilon) \in \mathbb{N}, \forall n > m > N(\varepsilon) \Rightarrow |u_n - u_m| < \frac{1}{2^m} < \varepsilon.$$

Example 3.15. *The sequence* $((-1)^n)_{n\in\mathbb{N}}$ *is not Cauchy sequence. Remark that for* $\varepsilon = 1$

$$\forall N \in \mathbb{N}, \exists n = N+1, m = N+2, n > m > N$$

and

$$|u_{N+2} - u_{N+1}| = \left|(-1)^{N+2} - (-1)^{N+1}\right| = 2 > 1.$$

3.1 The Cauchy Criterion

Our difficulty in proving " $u_n \to \ell$ " is this: What is ℓ ? Cauchy saw that it was enough to show that if the terms of the sequence got sufficiently close to each other. then completeness will guarantee convergence.

Theorem 3.10. *Every Cauchy sequence is bounded* [$\mathbb{R}$ *or* $\mathbb{C}$].

Proof. $1 > 0$ so there exists N such that $m, n \geqslant N \implies |u_m - u_n| < 1$. So for $m \geqslant N$, $|u_m| \leqslant 1 + |u_N|$ by the Δ law. So for all m

$$|u_m| \leqslant 1 + |u_1| + |u_2| + \cdots + |u_N|.$$

□

Theorem 3.11. *Every convergent sequence is Cauchy.*

Proof. Let $u_n \to l$ and let $\varepsilon > 0$. Then there exists N such that

$$k \geqslant N \implies |u_k - l| < \varepsilon/2$$

For $m, n \geqslant N$ we have

$$|u_m - l| < \varepsilon/2$$
$$|u_n - l| < \varepsilon/2$$

So

$$\begin{aligned} |u_m - u_n| &\leqslant |u_m - l| + |u_n - l| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon \end{aligned} \quad \text{by the } \Delta \text{ law}$$

□

Theorem 3.12. *Every real Cauchy sequence is convergent.*

Proof. Let the sequence be (u_n). By the above, (u_n) is bounded. By Bolzano-Weierstrass (u_n) has a convergent subsequence $(u_{n_k}) \to l$, say. So let $\varepsilon > 0$. Then

$$\exists N_1 \text{ such that } \quad r \geqslant N_1 \Longrightarrow |u_{n_r} - l| < \varepsilon/2$$

$$\exists N_2 \text{ such that } m, n \geqslant N_2 \Longrightarrow |u_m - u_n| < \varepsilon/2$$

Put $s := \min\{r \mid n_r \geqslant N_2\}$ and put $N = n_s$. Then

$$m, n \geqslant N \Longrightarrow |u_n - l| = |u_n - u_{n_s} + u_{n_s} - l| \leqslant |u_n - u_{n_s}| + |u_{n_s} - l| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}.$$

□

Theorem 3.13. *Every complex Cauchy sequence is convergent.*

Proof. Put $z_n = x_n + \mathrm{i}y_n$. Then x_n is Cauchy: $|x_x - x_m| \leqslant |z_n - z_m|$ (as $|Re(w)| \leqslant |w|$). So $x_n \to x, y_n \to y$ and so $z_n \to x + \mathrm{i}y$. □

Example 3.16. *Using Cauchy criterion, show that the sequence*

$$u_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n} \, (n \geqslant 1)$$

diverges.
Let $\varepsilon \in \left]0, \frac{1}{2}\right[$. Then for $n \geqslant 1$:

$$\begin{aligned} |u_{2n} - u_n| &= \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n} \\ &\geqslant \frac{1}{2n} + \frac{1}{2n} + \ldots + \frac{1}{2n} = n\frac{1}{2n} = \frac{1}{2} > \varepsilon \end{aligned}$$

So the Cauchy criterion is not satisfied.

3.2 Limit supremum and limit infimum

3.2.1 Short version

3.2.2 Generalization of the notion of the limit

Definition 3.9. *A number a is said to be an adherent point (also closure point or point of closure or contact point) of a sequence $\{u_n\}$ if there exists a subsequence of $\{u_n\}$ that converges to a.*

Example 3.17. *Let us consider the sequence* $\{u_n\}$ *defined by* $u_n = (-1)^n \left(1 + \frac{1}{n}\right) (n \in \mathbb{N})$.

Upper limit, lower limit:

$\overline{\lim} u_n = \sup Ad\{u_n\}$, $\underline{\lim} u_n = \inf Ad\{u_n\}$.

Note that $\overline{\lim} u_n$ and $\underline{\lim} u_n$ exist always in $\overline{\mathbb{R}}$ whatever the sequence $\{u_n\}$.

Moreover if the sequence is convergent then $\lim_{n\longrightarrow\infty} u_n = l \Longleftrightarrow \overline{\lim} u_n = \underline{\lim} u_n = l$.

Example 3.18. *Let the sequence* $\{u_n\}$ *be such that* $\lim_{n\longrightarrow\infty} u_n = l$. *Then* $Ad\{u_n\} = \{l\}$ *and consequently* $\overline{\lim} u_n = \underline{\lim} u_n = l$.

Example 3.19. *For the sequence* $\{u_n = n, n \in \mathbb{N}\}$, *we have* $Ad\{u_n\} = \{+\infty\}$. *So* $\overline{\lim} u_n = \underline{\lim} u_n = +\infty$.

Example 3.20. *Let the sequence* $\{(-1)^n, n \in \mathbb{N}\}$. *Then* $Ad\{u_n\} = \{-1, 1\}$. *so* $\overline{\lim} u_n = \sup\{-1, 1\} = 1$, $\underline{\lim} u_n = \inf\{-1, 1\} = -1$.

3.2.3 Long version

Let $(s_n)_n$ be a sequence of real numbers and define the sequences

$$u_k = \sup\{s_k, s_{k+1}, s_{k+2}, \ldots\} = \sup_{n\geq k} s_n$$
$$l_k = \inf\{s_k, s_{k+1}, s_{k+2}, \ldots\} = \inf_{n\geq k} s_n$$

For a simple example, consider the sequence $s_n = 1/n$. Then for each index k

$$u_k = \sup\left\{\frac{1}{k}, \frac{1}{k+1}, \frac{1}{k+2}, \ldots\right\} = \frac{1}{k}$$

because $1/k$ has the smallest denominator of all the fractions inside the braces, thus it must be the largest fraction. On the other hand,

$$l_k = \inf\left\{\frac{1}{k}, \frac{1}{k+1}, \frac{1}{k+2}, \ldots\right\} = 0$$

because the fractions inside the braces get smaller and smaller, approaching 0 .

For another example, consider the alternating sequence $s_n = (-1)^n$. In this case,

$$u_k = \sup\left\{(-1)^k, (-1)^{k+1}, (-1)^{k+2}, \ldots\right\} = 1$$

because the numbers inside the braces are always either -1 or 1 . Similarly,

$$l_k = \inf\left\{(-1)^k, (-1)^{k+1}, (-1)^{k+2}, \ldots\right\} = -1$$

Here is one more example that in a sense combines the previous two: $s_n = (-1)^n/n$. For every index k

$$u_k = \sup\left\{\frac{(-1)^k}{k}, \frac{(-1)^{k+1}}{k+1}, \frac{(-1)^{k+2}}{k+2}, \ldots\right\} = \begin{cases} 1/k & \text{if } k \text{ is even} \\ 1/(k+1) & \text{if } k \text{ is odd} \end{cases}$$

while

$$l_k = \inf\left\{\frac{(-1)^k}{k}, \frac{(-1)^{k+1}}{k+1}, \frac{(-1)^{k+2}}{k+2}, \ldots\right\} = \begin{cases} -1/(k+1) & \text{if } k \text{ is even} \\ -1/k & \text{if } k \text{ is odd} \end{cases}$$

There are also sequences for which (u_k) or (l_k) may be equal to plus or minus infinity. Consider the simple sequence $s_n = 2n$ for which we get

$$u_k = \sup\{2k, 2k+2, 2k+4, \ldots\}$$

The supremum equals plus infinity because the set in braces has no finite upper bounds. On the other hand,

$$l_k = \inf\{2k, 2k+2, 2k+4, \ldots\} = 2k$$

is well-defined for all k.

The following lists the basic properties of the two sequences (u_k) and (l_k).

Lemma 3.14. *Let (s_n) be a given sequence of real numbers.*

(a) The sequences (u_k) and (l_k) bound the sequence (s_n) in the following sense:

$$l_k \leq s_k \leq u_k$$

(b) (u_k) is a nonincreasing sequence, and (l_k) is a nondecreasing sequence.

Proof. (a) is clear from the definition of supremum and infimum of sets.

(b), we show that for every index k

$$u_k \geq u_{k+1} \quad \text{and} \quad l_k \leq l_{k+1}$$

For the first inequality, recall that for every k,

$$u_k = \sup\{s_k, s_{k+1}, s_{k+2}, \ldots\}$$
$$u_{k+1} = \sup\{s_{k+1}, s_{k+2}, s_{k+3}, \ldots\}$$

The only difference between the two quantities is that the second set doesn't contain (s_k). If (s_k) is less than or equal to one of the other numbers $s_{k+1}, s_{k+2}, \ldots$ inside the braces, then the supremum isn't affected by dropping it, and we have $u_{k+1} = u_k$. But if s_k is greater than all the other numbers inside the braces, then dropping it will reduce the supremum: $u_{k+1} < u_k$. $\square$

The previous Lemma shows that the bounding sequences u_k and l_k are monotone sequences. As such, each can either have a real number for a limit or diverge to ∞ or $-\infty$. Because u_k is nonincreasing, if its limit is a real number, then it must be the greatest lower bound or infimum of the sequence u_k and can thus be represented as

$$\lim_{k\to\infty} u_k = \inf_{k\geq 1} u_k = \inf_{k\geq 1} \sup\{s_k, s_{k+1}, s_{k+2}, \ldots\} = \inf_{k\geq 1} \sup_{n\geq k} s_n$$

Similarly, for l_k, which is nondecreasing, we can write

$$\lim_{k\to\infty} l_k = \sup_{k\geq 1} l_k = \sup_{k\geq 1} \inf\{s_k, s_{k+1}, s_{k+2}, \ldots\} = \sup_{k\geq 1} \inf_{n\geq k} s_n$$

If the limits are $\pm\infty$ instead of real numbers, then we use those symbols to indicate the limits. With this in mind, we have the following definition.

Let s_n be a given sequence of real numbers. If the sequence u_k converges to a real number, then its limit is the limit supremum (or limit superior) of s_n and denoted by

$$\limsup_{n\to\infty} s_n = \inf_{n\geq 1} \sup\{s_n, s_{n+1}, s_{n+2}, \ldots\} = \overline{lim}_{k\to\infty} u_k$$

If u_k diverges to ∞ or $-\infty$, then we use these symbols to denote the limit supremum. Similarly, the limit infimum (or limit inferior) of s_n is

$$\liminf_{n\to\infty} s_n = \sup_{n\geq 1} \inf\{s_n, s_{n+1}, s_{n+2}, \ldots\} = \underline{lim}_{k\to\infty} l_k$$

or ∞ or $-\infty$ as appropriate.

For example, referring to the example we discussed earlier, we have for $s_n = 1/n$

$$\limsup_{n\to\infty} \frac{1}{n} = \overline{\lim}_{k\to\infty} \frac{1}{k} = 0, \quad \liminf_{n\to\infty} \frac{1}{n} = \underline{lim}_{k\to\infty} 0 = 0$$

Similarly, for $s_n = (-1)^n$

$$\limsup_{n\to\infty} (-1)^n = \overline{\lim}_{k\to\infty} 1 = 1, \quad \liminf_{n\to\infty} (-1)^n = \underline{\lim}_{k\to\infty} (-1) = -1$$

And for $s_n = 2n$

$$\limsup_{n\to\infty} (2n) = \infty, \quad \liminf_{n\to\infty} (2n) = \lim_{k\to\infty} 2k = \infty$$

Notice that of the above three sequences, only $1/n$ converges to a real number, and it has the property that its limit supremum and limit infimum are equal real numbers.

Theorem 3.15. *A sequence (s_n) converges to a real number s if and only if*

$$\limsup_{n\to\infty} s_n = \liminf_{n\to\infty} s_n = s$$

Proof. First, we assume that $\limsup_{n\to\infty} s_n = \liminf_{n\to\infty} s_n = s$ is given and prove that s_n must converge to the number s. By

$$l_n \leq s_n \leq u_n$$

Since $\lim_{n\to\infty} l_n = s$ and also $\lim_{n\to\infty} u_n = s$, the squeeze theorem implies that $\lim_{n\to\infty} s_n$ exists and equals s.

Conversely, assume that s_n converges to a number s. Then by the definition of convergence, for every $\varepsilon > 0$, we can find a positive integer N such that

$$|s_n - s| < \varepsilon \quad \text{for all } n \geq N$$

or equivalently,

$$s - \varepsilon < s_n < s + \varepsilon \quad \text{for all } n \geq N$$

In particular, $s_{n+j} < s + \varepsilon$ for all $j = 1, 2, 3, \ldots$ and $n \geq N$, and this implies that

$$\sup\{s_n, s_{n+1}, s_{n+2}, \ldots\} \leq s + \varepsilon \quad \text{for all } n \geq N$$

because $s + \varepsilon$ is an upper bound of the set $\{s_n, s_{n+1}, s_{n+2}, \ldots\}$, while the supremum is its least upper bound.

Similarly, because $s_{n+j} > s - \varepsilon$ for all j, we infer that $s - \varepsilon$ is a lower bound for $\{s_n, s_{n+1}, s_{n+2}, \ldots\}$, and therefore, the greatest lower bound of this set satisfies

$$\inf\{s_n, s_{n+1}, s_{n+2}, \ldots\} \geq s - \varepsilon \quad \text{for all } n \geq N$$

From above, we conclude that

$$s - \varepsilon \leq l_n \leq u_n \leq s + \varepsilon \quad \text{for all } n \geq N$$

These inequalities imply the following:

$$|u_n - s| \leq \varepsilon, \quad |l_n - s| \leq \varepsilon \quad \text{for all } n \geq N$$

Since these inequalities hold for all $\varepsilon > 0$, we conclude that

$$\lim_{n\to\infty} u_n = s \text{ and } \lim_{n\to\infty} l_n = s.$$

$\square$

An immediate consequence of the above theorem is the following:

Corollary 3.3. *If* $\limsup_{n\to\infty} s_n \neq \liminf_{n\to\infty} s_n$ *for a sequence* (s_n)*, then* (s_n) *diverges.*

Notice that the above corollary includes the cases where limit supremum or limit infimum are ∞ or $-\infty$. For example, the previous Corollary implies the divergence of both of the sequences $(-1)^n$ and $2n$ that we discussed earlier. A sequence (s_n) can have a limit s only when the upper and lower bounding sequences meet:

$$l_1 \leq l_2 \leq \cdots \leq l_k \leq \cdots \to s \leftarrow \cdots \leq u_k \leq \cdots \leq u_2 \leq u_1$$

If the lower sequence does not meet the upper one, then there is a nonempty open interval of numbers (l, u) between them:

$$l_1 \leq l_2 \leq \cdots \to \lim_{n\to\infty} l_n = l < u = \lim_{n\to\infty} u_n \leftarrow \cdots \leq u_2 \leq u_1$$

The sequence s_n cannot converge to a limit if the interval (l, u) is not empty, i.e., $u - l > 0$ because not matter how large the index N we choose, there are terms $s_k \leq l$ and other terms $s_m \geq u$ with $k, m \geq N$; so if we choose, say, $\varepsilon = (u - l)/2$, then no valid threshold index N can be found to match such values of ε.

Every sequence (s_n) has a limit supremum u and a limit infimum l (they could be ∞ or $-\infty$ if (s_n) is unbounded). Although u and l are limits of monotone sequences (u_k) and (l_k) that are derived from (s_n), these bounding sequences may or may not contain terms of s_n; in fact, there are sequences where $u_k \neq s_n$ and $l_k \neq s_n$ for every k and every n. On the other hand, the definitions of the bounding sequences suggest that the numbers (u_k) and (l_k) are increasingly aligned with the terms of (s_n) as k and n get larger. This raises a natural question: are there subsequences of (s_n) that converge to the limits u and l ?

To answer this question, let (s_n) be a given sequence and consider its upper bounding sequence

$$u_k = \sup \{s_k, s_{k+1}, s_{k+2}, \ldots\}$$

If u is the limit supremum of s_n, then because u_k is a nonincreasing sequence,

$$u = \inf_{k\geq n} u_k = \lim_{k\to\infty} u_k$$

If we pick any number $\varepsilon > 0$, then there is a positive integer N such that

$$u_k - u = |u_k - u| < \varepsilon \quad \text{for all } k \geq N$$

Further, $u_k \geq s_n$ for all $n \geq k$ so that

$$s_n \leq u_k < u + \varepsilon \quad \text{for all } n \geq N$$

Next, since u_k is the least upper bound of the set $\{s_k, s_{k+1}, s_{k+2}, \ldots\}$ for each k and further, $u \leq u_k$ for all k, it follows that $u - \varepsilon$ is not an upper bound of this set. This means that there is an integer $m \geq 0$ such that $u - \varepsilon < s_{k+m}$. If $k \geq N$, then

$$u - \varepsilon < s_{k+m} < u + \varepsilon \quad k \geq N$$

These inequalities help us identify a subsequence of s_n that converges to u.

Theorem 3.16. *The following statements are equivalent*
(a) If A has a supremum or least upper bound, then it is unique. Also, a greatest lower bound is unique, if it exists. (b) There is a sequence a_n in A that converges to sup A. Also there is a sequence in A that converges to $\inf A$.

Proof. a) Let $r = \sup A$. If r' is also a least upper bound of A, then in particular, r' is an upper bound, so $r \leq r'$. Similarly, $r' \leq r$ since r is also an upper bound of A, and r' is least by assumption. It follows that $r' = r$. The proof that the greatest lower bound is unique is essentially the same.

(b) We prove the assertion about $\sup A$ and leave the one about $\inf A$ as an exercise. First, if $s = \sup A$ is in A (e.g., if A is a finite set), then the constant sequence $a_n = s$ converges to s (trivially), and the proof is finished. Next, suppose that s is not in A (hence, A is infinite). If $\sup A = \infty$, then for every positive integer n there is an element a_n of S such that $a_n \geq n$. It follows that a_n diverges to ∞ and thus to $\sup A$. Finally, let $\sup A = s < \infty$. Then there is an element of $a_1 \in A$ such that $s - a_1 < 1$; if not, then $s \geq a+1$ for all $a \in A$, and thus s is not the least upper bound. Therefore, $s-1 < a_1 < s$ Similarly, there is $a_2 \in A$ such that $a_2 > s - 1/2$ and so on; for every n, there is $a_n \in A$ such that $s - \frac{1}{n} < a_n < s$ Since $\lim_{n\to\infty}(s - 1/n) = s$, the squeeze theorem implies that $\lim_{n\to\infty} a_n = s$. □

3.3 The Stolz-Cesaro Theorem

Theorem 3.17. *If $(b_n)_n$ is a sequence of positive real numbers, such that $\sum_{n=1}^{\infty} b_n = \infty$, then for any sequence $(a_n)_n \subset \mathbb{R}$ one has the inequalities:*

$$\limsup_{n\to\infty} \frac{a_1 + a_2 + \cdots + a_n}{b_1 + b_2 + \cdots + b_n} \leq \limsup_{n\to\infty} \tfrac{a_n}{b_n} \tag{3.2}$$

$$\liminf_{n\to\infty} \frac{a_1 + a_2 + \cdots + a_n}{b_1 + b_2 + \cdots + b_n} \geq \liminf_{n\to\infty} \tfrac{a_n}{b_n}. \tag{3.3}$$

In particular, if the sequence $(a_n/b_n)_n$ has a limit, then

$$\lim_{n\to\infty} \frac{a_1 + a_2 + \cdots + a_n}{b_1 + b_2 + \cdots + b_n} = \lim_{n\to\infty} \frac{a_n}{b_n}.$$

Proof. It is quite clear that we only need to prove (3.2), since the other inequality follows by replacing a_n with $-a_n$.

The inequality (3.2) is trivial, if the right-hand side is $+\infty$. Assume then that the quantity $L = \limsup_{n\to\infty} (a_n/b_n)$ is either finite or $-\infty$, and let us fix for the moment some number $\ell > L$. By the definition of limsup, there exists some index $k \in \mathbb{N}$, such that

$$\frac{a_n}{b_n} \leq \ell, \quad \forall n > k. \tag{3.4}$$

Using (3.4) we get the inequalities

$$a_1 + a_2 + \cdots + a_n \leq a_1 + \cdots + a_k + \ell\left(b_{k+1} + b_{k+2} + \ldots b_n\right), \quad \forall n > k.$$

If we denote for simplicity the sums $a_1 + \cdots + a_n$ by A_n and $b_1 + \cdots + b_n$ by B_n, the above inequality reads:

$$A_n \leq A_k + \ell\left(B_n - B_k\right), \quad \forall n > k,$$

so dividing by B_n we get

$$\frac{A_n}{B_n} \leq \ell + \frac{A_k - \ell B_k}{B_n}. \tag{3.5}$$

Since $B_n \to \infty$, by fixing k and taking limsup in (3.5), we get $\limsup_{n\to\infty} (A_n/B_n) \leq \ell$. In other words, we obtained the inequality

$$\limsup_{n\to\infty} \frac{a_1 + \cdots + a_n}{b_1 + \cdots + b_n} \leq \ell, \quad \forall \ell \geq L,$$

which in turn forces

$$\limsup_{n\to\infty} \frac{a_1 + \cdots + a_n}{b_1 + \cdots + b_n} \leq L$$

□

Remark. An equivalent formulation of the above Theorem is as follows: If $(y_n)_n$ is a strictly increasing sequence with $\lim_{n\to\infty} y_n = \infty$, then for any sequence $(x_n)_n$, the following inequalities hold:

$$\limsup_{n\to\infty} \frac{x_n}{y_n} \leq \limsup_{n\to\infty} \frac{x_n - x_{n-1}}{y_n - y_{n-1}}$$
$$\liminf_{n\to\infty} \frac{x_n}{y_n} \geq \liminf_{n\to\infty} \frac{x_n - x_{n-1}}{y_n - y_{n-1}}$$

In particular, if the sequence $\left(\frac{x_n - x_{n-1}}{y_n - y_{n-1}}\right)_n$ has a limit, then

$$\lim_{n\to\infty} \frac{x_n}{y_n} = \lim_{n\to\infty} \frac{x_n - x_{n-1}}{y_n - y_{n-1}}.$$

Indeed (assuming all the y_n 's are positive, which happens anyway for n large enough), if we consider the sequences $(a_n)_n$ and $(b_n)_n$, defined by $a_1 = x_1, b_1 = y_1$, and $a_n = x_n - x_{n-1}$, $b_n = y_n - y_{n-1}, n \geq 2$, then everything is clear, since $x_n = a_1 + \cdots + a_n$ and $y_n = b_1 + \cdots + b_n$.

The Stolz-Cesaro Theorem has numerous applications in Calculus. Below are three of the most significant ones.

Theorem 3.18. *Cesaro's Theorem*
For any sequence $(a_n)_n \subset \mathbb{R}$ one has the inequalities:

$$\limsup_{n\to\infty} \frac{a_1 + a_2 + \cdots + a_n}{n} \leq \limsup_{n\to\infty} a_n$$
$$\liminf_{n\to\infty} \frac{a_1 + a_2 + \cdots + a_n}{n} \geq \liminf_{n\to\infty} a_n.$$

In particular, if the sequence $(a_n)_n$ has a limit, then

$$\lim_{n\to\infty} \frac{a_1 + a_2 + \cdots + a_n}{n} = \lim_{n\to\infty} a_n.$$

Proof. Particular case of Stolz-Cesaro Theorem with $b_n = 1$. □

Remark 3.7. *An equivalent formulation of the above Theorem (proven using the alternative version of Stolz-Cesaro Theorem) is as follows: For any sequence $(x_n)_n$, the following inequalities hold:*

$$\limsup_{n\to\infty} \frac{x_n}{n} \leq \limsup_{n\to\infty} (x_n - x_{n-1})$$
$$\liminf_{n\to\infty} \frac{x_n}{n} \geq \liminf_{n\to\infty} (x_n - x_{n-1})$$

In particular, if the sequence $(x_n - x_{n-1})_n$ has a limit, then

$$\lim_{n\to\infty} \frac{x_n}{n} = \lim_{n\to\infty} (x_n - x_{n-1}) .$$

3.4 Sequences defined by recursion formulas

Definition 3.10. *Let $f : D \subset \mathbb{R} \to \mathbb{R}$. A recursive sequence is a sequence in which terms are defined using one or more previous terms which are given by $u_0 \in D$ and the relation $\forall n \in \mathbb{N} : u_{n+1} = f(u_n)$*

We suppose that $f(D) \subset D$, and so the sequence is well defined.

Example 3.21. *Let $u_{n+1} = \sqrt{u_n + 2}$, $u_0 = 1$, we have $f(x) = \sqrt{x+2}, D = [-2, +\infty[$ et $f(D) = [0, +\infty[\subset D$. Thus u_n is well defined.*

Lemma 3.19. *If f is continuous on D and the sequence $\{u_n\}$ converges to $l \in D$, then $l = f(l)$.*

Theorem 3.20. *Let $\{u_n\}$ be a real sequence recursively defined by $u_{n+1} = f(u_n)$.*

(a) If f is nondecreasing then the sequence (u_n) is monotone. More precisely (u_n) is monotone
1) If $u_0 \leqslant u_1$, then the sequence is nondecreasing.
2) If $u_0 \geqslant u_1$, then the sequence is nonincreasing.

(b) If f is nonincreasing then the sequence $u_{n+1} \leqslant u_n$, is positive and negative alternatively. We set $g = fof$, so g is nondecreasing, we can easily verify that the sequences (u_{2n}) and (u_{2n+1}) defined by $u_{2n+2} = f(f(u_{2n}))$, $u_2 = f(u_1)$ and $u_{2n+1} = f(f(u_{2n-1}))$, u_1 given, are oppositely monotone such as that $g(u_1) - u_1 = f(f(u_1)) - u_1$ and $g(f(u_1)) - f(u_1) = f(f(f(u_1))) - f(u_1)$ have different signs.

CHAPTER

4

REAL FUNCTIONS OF REAL VARIABLES

4.1 Introduction

Let $D \in \mathbb{R}$ i.e D is a subset of the real numbers. Often we need to associate with $x \in D$ a new real number which we denote at the moment by $f(x)$. For example the absolute value.

Definition 4.1. *Let $D \in \mathbb{R}$. A function $f : D \longmapsto R$ is a rule which assigns to every $x \in D$ exactly one real value $f(x)$. For this we write $x \longmapsto f(x)$ and say that x is mapped onto $f(x)$, or $f(x)$ is the value of f at x.*

We call D the domain of the function f, sometimes we write $D(f)$ or D_f instead of D. So $D_f = \{x \in D : f(x) \text{ exists}\}$.

The set of all functions is denoted by $\mathcal{F}(D, \mathbb{R})$.

Definition 4.2. *The set $\{y = f(x), x \in D\}$ is called the range of f and is denoted by $H(f)$. Variable x is called argument or independent variable and variable y is called dependent.*

Definition 4.3. *Let* $f, g \in \mathcal{F}(D, \mathbb{R})$ *and* $\lambda \in \mathbb{R}$. *We define the following important operations:*

1. $(f+g)(x) = f(x) + g(x)$ $(\forall x \in D)$
2. $(f.g)(x) = f(x).g(x)$ $(\forall x \in D)$
3. $(\lambda g)(x) = \lambda g(x)$ $(\forall \lambda \in \mathbb{R}, \forall x \in D)$
4. *If* $\forall x \in D : f(x) \neq 0$, $\left(\frac{1}{f}\right)(x) = \frac{1}{f(x)}$.

Definition 4.4. *Let* f *and* g *be real functions with domains* $D(f)$ *and* $D(g)$ *. Let* $H(f) \subset D(g)$. *Then under the composition of function* f *and* g *we understand function* h *defined by* $\forall x \in D(h) : h(x) = g(f(x))$, *with D(h) = D(f).*

Notation: $h = gof$.

Definition 4.5. *Two functions* f *and* g *are equal* $(f = g)$, *if*
(i) $D(f) = D(g)$
(ii) $\forall x \in D(f) : f(x) = g(x)$

Remark 4.1. *In general* $gof \neq fog$.

Definition 4.6. *Graph of function* f *is a set of ordered pairs of real numbers* $(x, f(x))$, *where* $x \in D(f)$. *We write*

$$graphf = \{(x, f(x))/x \in D(f)\}$$

Even and odd functions

Definition 4.7. *Let* $D \subset \mathbb{R}$ *such that* $(\forall x \in D) \Longrightarrow (-x \in D)$
We say that function $f : D \longrightarrow \mathbb{R}$ *is even if and only if* $(\forall x \in D), (f(-x) = f(x))$.
We say that function $f : D \longrightarrow \mathbb{R}$ *is odd if and only if* $(\forall x \in D), (f(-x) = -f(x))$.

Periodic functions

Definition 4.8. *A function* $f : D \longrightarrow \mathbb{R}$ *is called periodic if* $\exists \quad T > 0$ *such that*
– $\forall x \in D \Longrightarrow x \pm T \in D, f(x+T) = f(x)$
– $\forall x \in D, f(x+T) = f(x)$. *Number* T *is called a period of* f. *The smallest positive period is called primitive.*

Remark 4.2. *Let* $T > 0$ *be a period of* f, *then*

$$\forall x \in \mathbb{R}, \forall n \in \mathbb{Z} : f(x + nP) = f(x).$$

Theorem 4.1. *(i) If f is periodic with period P and function g such that $H(f) \subset D(g)$ then a composition $h(x) = g(f(x))$ is periodic with the same period p. (ii) If f is periodic with period p and $a \in \mathbb{R}, a \neq 0$; then function $g(x) = f(ax)$ is periodic with period $\frac{P}{a}$.*

Proof. Just do it. $\square$

Bounded Functions

Definition 4.9. *Let $f : D \to \mathbb{R}$ and let $f(D)$ the set of all the values of f.*

We say that function f is bounded above on its domain D if $f(D)$ is bounded above i.e. $\exists M \in \mathbb{R}, \forall x \in D : f(x) \leqslant M$.
We say that function f is bounded below on its domain D if $f(D)$ is bounded below i.e. $\exists m \in \mathbb{R}, \forall x \in D : f(x) \geqslant m$.
We say that function f is bounded on its domain D if $f(D)$ is bounded i.e. $\exists A \in \mathbb{R}, \forall x \in D : |f(x)| \leqslant A$.

Remark 4.3. *If f is bounded on D, then $f(D)$ admits a supremum M and an infimum m. We denote $M = \sup\limits_{x \in D} f(x)$, and $m = \inf\limits_{x \in D} f(x)$. We have*

$$\sup_{x \in D} f(x) = M < +\infty \Longleftrightarrow \begin{cases} \forall x \in D : f(x) \leqslant M \\ \forall \varepsilon > 0, \exists x_0 \in D : f(x_0) > M - \varepsilon \end{cases}$$

$$\inf_{x \in D} f(x) = m > -\infty \Longleftrightarrow \begin{cases} \forall x \in D : f(x) \geqslant m \\ \forall \varepsilon > 0, \exists x_0 \in D : f(x_0) < m + \varepsilon \end{cases}$$

Monotone functions

Definition 4.10. *Consider $f : D \subset \mathbb{R} \to \mathbb{R}$, and set $M \subset D$.*

1) f is nondecreasing on $M \Leftrightarrow \forall x_1, x_2 \in M,\ x_1 \leqslant x_2 \Longrightarrow f(x_1) \leqslant f(x_2)$

2) $f$is nonincreasing on $M \Leftrightarrow \forall x_1, x_2 \in M,\ x_1 \leqslant x_2 \Longrightarrow f(x_1) \geqslant f(x_2)$

3) f is increasing on $M \Leftrightarrow \forall x_1, x_2 \in M,\ x_1 < x_2 \Longrightarrow f(x_1) < f(x_2)$

4) f is decreasing on $M \Leftrightarrow \forall x_1, x_2 \in M,\ x_1 < x_2 \Longrightarrow f(x_1) > f(x_2)$

Definition 4.11. *If f satisfies any of condition $(1-4)$ we call it monotone. If f has property (3) or (4), we call it strictly monotone.*

Corollary 4.1. *We can check that in case where f is increasing or decreasing then we have $x_1 \neq x_2 \Longrightarrow f(x_1) \neq f(x_2)$, so f is injective.*
A sum of two increasing (decreasing) functions is an increasing (decreasing) function.

Inverse functions

Definition 4.12. *Let $f : D(f) \longleftrightarrow \mathbb{R}$ be an injective function with range $H(f)$. Inverse function of f (denoted f^{-1}) is defined by the relation $y = f(x) \Longleftrightarrow x = f^{-1}(y)$. Obviously the domain $D(f^{-1}) = H(f)$ and range $H(f^{-1}) = D(f)$.*

Remark 4.4. *(i) Graph of f^{-1} is symmetric to the graph of f with respect to a line $y = x$.*
(ii) $\forall x \in D(f) : f^{-1}(f(x)) = x$.
(iii)$\forall y \in D(f^{-1}) = H(f) : f(f^{-1}(y)) = y$.
(iv) $(f^{-1})^{-1} = f$.

Lemma 4.2. *Let A, B be two subset of $\mathbb{R}$. Let $f : A \longrightarrow B$ be a bejective and strictly monotonic function. The f^{-1} is strictly monotonic function, the same monotonic as f.*

Proof. WLOG, we can suppose that f is strictly increasing. let y_1, y_2 be two elements of B such that $y_1 < y_2$, then we prove that $f^{-1}(y_1) < f^{-1}(y_2)$. Since f is bejective then there exist x_1, x_2 such that $f(x_1) = y_1 \quad \text{and} \quad f(x_2) = y_2$. Let us proceed by contra-positive, we suppose that $x_1 \geq x_2$ then $y_1 = f(x_1) \geq f(x_2) = y_2$ which is a contradiction with $y_1 < y_2$. □

Lemma 4.3. *Let I be an interval of $\mathbb{R}$ and $f : I \longrightarrow \mathbb{R}$ be a strictly monotonic function such that $f(I)$ is an interval, then f is necessary continuous on I.*

Proof. WLOG, we can suppose that I is not a trivial interval $I = \emptyset, \quad I = \{a\}$ and we suppose that f is strictly increasing. Let $a \in I$, if $a > inf(I)$, we prove that $\lim\limits_{x \to a^-} f(x) = f(a)$ and if $a < sup(I)$, we prove that $\lim\limits_{x \to a^+} f(x) = f(a)$. Indeed let $a \in I$, if $a > inf(I)$, by monotonic limit theorem, there exists a real l such that $\lim\limits_{x \to a^-} f(x) = l$, and $l \leq f(a)$. Actually, $l = sup\{f(x), x \in I, x < a\}$, we aim to prove that $l = f(a)$. By contrapositive, suppose that $l < f(a)$, then there exists $l < m < f(a)$, there exists $\alpha \in I$ such that $\alpha < a$. By the hypothesis that f is increasing, we get $f(\alpha) \leq l < m < f(a)$. $f(I)$ is an interval, so there exists $c \in I$ such that $f(c) = m$.

- If $c \geq a$, and f is supposed to be increasing, hence $f(c) \geq f(a) > m$,
- If $c < a$, monotonic limit gives $f(c) < l < m$, which is a contradiction, thus $f(a) = l$.

Similarly, with the right limit. □

Lemma 4.4. *Let I be an interval of $\mathbb{R}$ and $f : I \longrightarrow \mathbb{R}$ be a continuous and injective function. Then f is strictly monotonic.*

Proof. By contrapositive principal, we suppose that f isn't strictly monotone, thus

- $\exists x, y \in I : x < y$ and $f(x) \geq f(y)$
- $\exists x', y' \in I : x' < y'$ and $f(x') \leq f(y')$

The segments $[x, x'], [y, y']$ are defined by $[x, x'] = \{tx + (1-t)x', t \in [0,]\}$ and $[y, y'] = \{ty + (1-t)y', t \in [0, 1]\}$. Then let us define the following functions:

$$\alpha : [0, 1] \longrightarrow \mathbb{R} : t \to tx + (1-t)x',$$

$$\beta : [0, 1] \longrightarrow \mathbb{R} : t \to ty + (1-t)y'.$$

$\alpha(t)$ and $\beta(t)$ belong to the interval I. Now we consider the function defined by

$$\phi : [0, 1] \longrightarrow \mathbb{R} : t \to f(\alpha(t)) - f(\beta(t)).$$

(1) α, β, f are continuous so ϕ,
(2) $\phi(0) = f(x) - f(y) \geq 0$ and $\phi(1) = f(x') - f(y') \leq 0$. The mean theorem value implies that there exists $t_0 \in]0, 1[$ such that $\phi(t_0) = 0$, which means that $f(\alpha(t_0) = f(\beta(t_0))$. Contradiction (f is injective). □

The inverse function theorem for strictly monotonic function

Theorem 4.5. *Let I be an interval of $\mathbb{R}$ and $f : I \longrightarrow \mathbb{R}$ be a function. Set $J = f(I)$. Then two of the following properties implie the third one.*

1- J is an interval and $f : I \longrightarrow J$ is a bejection function.

2- f is strictly monotonic on I.

3- f is continuous on I.

more; if 1, 2 and 3 are satisfied, then the inverse function $f^{-1} : J \longrightarrow I$ is continuous, strictly monotonic, the same as f.

Proof.
- If 1 and 2 are satisfied then f is continuous (Lemma (4.26)).
- If 1 and 3 are satisfied then f is strictly monotonic (Lemma (4.27)).
- If 2 and 3 are satisfied then J is an interval (MTV theorem). f is strictly monotonic, thus f is injective and by the way bejective.

If 1, 2 and 3 are satisfied the by (Lemma (4.25)), $f^{-1} : J \longrightarrow I$ is strictly monotonic, the same as f. $f^{-1} : J \longrightarrow I$ realize a bejection from J on I, so f^{-1} satisfies 1 and 2, hence f^{-1} is continuous. □

Inverse image

Let $f : A \longrightarrow B$ be a function, and let $U \subset B$ be a subset. The inverse image (or, preimage) of U is the set $f^{-1}(U) \subset A$ consisting of all elements $a \in A$ such that $f(a) \in U$.

The inverse image commutes with all set operations: For any collection $\{U_i\}_{i\in I}$ of subsets of B, we have the following identities for

(1) Unions:

$$f^{-1}\left(\bigcup_{i\in I} U_i\right) = \bigcup_{i\in I} f^{-1}\left(U_i\right)$$

(2) Intersections:

$$f^{-1}\left(\bigcap_{i\in I} U_i\right) = \bigcap_{i\in I} f^{-1}\left(U_i\right)$$

and for any subsets U and V of B, we have identities for

(3) Complements:

$$\left(f^{-1}(U)\right)^{\complement} = f^{-1}\left(U^{\complement}\right)$$

(4) Set differences:

$$f^{-1}(U \backslash V) = f^{-1}(U) \backslash f^{-1}(V)$$

(5) Symmetric differences:

$$f^{-1}(U \triangle V) = f^{-1}(U) \triangle f^{-1}(V)$$

In addition, for $X \subset A$ and $Y \subset B$, the inverse image satisfies the miscellaneous identities

(6) $\left(f|_X\right)^{-1}(Y) = X \cap f^{-1}(Y)$

(7) $f\left(f^{-1}(Y)\right) = Y \cap f(A)$

(8) $X \subset f^{-1}(f(X))$, with equality if f is injective. Let $f : A \longrightarrow B$ be a function, and let $U \subset A$ be a subset. The direct image (or, simply, image) of U is the set $f(U) \subset B$ consisting of all elements of B which equal $f(u)$ for some $u \in U$.

Direct images satisfy the following properties:

(1) Unions: For any collection $\{U_i\}_{i\in I}$ of subsets of A,

$$f\left(\bigcup_{i\in I} U_i\right) = \bigcup_{i\in I} f\left(U_i\right).$$

(2) Intersections: For any collection $\{U_i\}_{i\in I}$ of subsets of A,

$$f\left(\bigcap_{i\in I} U_i\right) \subset \bigcap_{i\in I} f\left(U_i\right).$$

(3) Set difference: For any $U, V \subset A$,

$$f(V \backslash U) \supset f(V) \backslash f(U).$$

In particular, the complement of U satisfies $f\left(U^{\complement}\right) \supset f(A) \backslash f(U)$.

(4) Subsets: If $U \subset V \subset A$, then $f(U) \subset f(V) \subset B$.

(5) Inverse image of a direct image: For any $U \subset A$,

$$f^{-1}(f(U)) \supset U$$

with equality if f is injective.

(6) Direct image of an inverse image: For any $V \subset B$,

$$f\left(f^{-1}(V)\right) \subset V$$

with equality if f is surjective.

Local maximum, local minimum Local maximum and minimum are the points of the functions, which give the maximum and minimum range. The local maxima and local minima can be computed by finding the derivative of the function. The first derivative test and the second derivative test are the two important methods of finding the local maximum and local minimum.

Definition 4.13. *Let $f : D \subset \mathbb{R} \to \mathbb{R}$ and $x_0 \in E$.*
– x_0 is said to be local maximum of f if there exists $\alpha > 0$ such that f is nondecreasing on $]x_0 - \alpha, x_0[$ and nonincreasing on $]x_0, x_0 + \alpha[$.
– x_0 is said to be local minimum of f if there exists $\alpha > 0$ such that f is nonincreasing on $]x_0 - \alpha, x_0[$ and nondecreasing on $]x_0, x_0 + \alpha[$.

Order relation on $\mathcal{F}(D, \mathbb{R})$.

Definition 4.14. *Let $f, g : D \subset \mathbb{R} \to \mathbb{R}$.*
1) f is said to be positive $f \geqslant 0$, if : $\forall x \in D : f(x) \geqslant 0$ (resp. negative, $f \leqslant 0$ if $\forall x \in D : f(x) \leqslant 0$.
2) f is said to be greater then $g, (f \geqslant g)$, if: $\forall x \in D : f(x) \geqslant g(x)$.
3) f is said to be smaller then $g, (f \leqslant g)$, if: $\forall x \in D : f(x) \leqslant g(x)$.

Remark 4.5. *We can easily check that this order relation isn't total.*

4.2 Limit of a Function

The basic idea underlying the concept of the limit of a function f at a point x_0 is to study the behaviour of f at points close to, but not equal to, x_0. We illustrate this with the following simple examples. Suppose that the velocity $v(m/s)$ of a falling object is given as a function $v = v(t)$ of time t. If the object hits the ground in $t = 2$, then $v(2) = 0$. Thus to find the velocity at the time of impact, we investigate the behaviour of $v(t)$ as t approaches 2, but is not equal to 2. Neglecting air resistance, the function $v(t)$ is given as follows :

$$v(t) = \begin{cases} 32t, 0 \leqslant t < 2, \\ 0, t \geqslant 2. \end{cases}$$

Our intuition should convince us that $v(t)$ approaches $64m/s$ as t approaches 2 , and that this is the velocity upon impact. As another example, consider the function $f(x) = xsin\left(\frac{1}{x}\right)$, $x \neq 0$. Here the function f is not defined at $x = 0$. Thus to investigate the behaviour of f at 0 we need to consider the values $f(x)$ for x close to, but not equal to 0 . Since

$$|f(x)| = \left|xsin\left(\frac{1}{x}\right)\right| \leqslant |x|$$

for all $x \neq 0$, our intuition again should tell us that $f(x)$ approaches 0 as x approaches 0. We now make this idea of (x) approaching a value L as x approaches a point x_0 precise. In order that the definition be meaningful, we must require that the point p be a limit point of the domain of the function f .

Definition 4.15. *Let $x_0 \in \mathbb{R}$ an accumulation point of a subset $D \neq \phi \subset \mathbb{R}$ and $f : D \mapsto \mathbb{R}$ a function defined on a neighbourhood of x_0 except may be at x_0. The function f has a limit at x_0 if there exists $l \in \mathbb{R}$ such that*

$$\forall \varepsilon > 0, \exists \delta\left(x_0, \varepsilon\right) > 0, \forall x : 0 < |x - x_0| < \delta \Rightarrow |f\left(x\right) - l| < \varepsilon$$

Customary we write: $\lim\limits_{x \to x_0} f\left(x\right) = l$ or $f\left(x\right) \to l,\ x \to x_0$
Shortly

$$\lim_{x \to x_0} f\left(x\right) = l \Leftrightarrow \forall \varepsilon > 0, \exists \delta\left(x_0, \varepsilon\right) > 0, \forall x : 0 < |x - x_0| < \delta \Rightarrow |f\left(x\right) - l| < \varepsilon$$

Uniqueness of the limit

Theorem 4.6. *If f admits a limit then it is unique.*

Proof. Just do it. □

One-Sided Limits

It is possible for a function to fail to have a limit at a point and yet appear to have limits on one side. If we ignore what is happening on the right for a function, perhaps it will have a " left-hand limit." This is easy to achieve

Definition 4.16. *We say that a function f has right-hand limit (resp. left-hand limit) at x_0 if $\forall\varepsilon > 0, \exists\delta > 0, \forall x : x_0 < x < x_0 + \delta$ (resp. $x_0 - \delta < x < x_0$) $\Rightarrow |f(x) - l| < \varepsilon$ we note*

$$\lim_{x\to x_0+0} f(x) = f(x_0 + 0), \lim_{x\to x_0-0} f(x) = f(x_0 - 0).$$

We have

$$\lim_{x\to x_0} f(x) = l \Longleftrightarrow \lim_{x\to x_0+0} f(x) = \lim_{x\to x_0-0} f(x) = l$$

Limit at infinity

Definition 4.17. *Let $x_0 = +\infty$ be an accumulation point of a given subset D. Then*

$$\lim_{x\to+\infty} f(x) = l \Leftrightarrow \forall\varepsilon > 0, \exists A > 0, \forall x > A \Rightarrow |f(x) - l| < \varepsilon$$

Let $x_0 = -\infty$ be an accumulation point of a given subset D. Then

$$\lim_{x\to-\infty} f(x) = l \Leftrightarrow \forall\varepsilon > 0, \exists A > 0, \forall x < -A \Rightarrow |f(x) - l| < \varepsilon$$

Infinite limits

Definition 4.18. *Let $x_0 \in \mathbb{R}$ be an accumulation point of a given subset D and $f : D \to \mathbb{R}$. Then*

$$\lim_{x\to x_0} f(x) = +\infty \Leftrightarrow \forall A > 0, \exists\delta > 0, \forall x : 0 < |x - x_0| < \delta \Rightarrow f(x) > A$$

$$\lim_{x\to x_0} f(x) = -\infty \Leftrightarrow \forall A > 0, \exists\delta > 0, \forall x : 0 < |x - x_0| < \delta \Rightarrow f(x) < -A$$

Main limit's theorems

Our first theorem allows us to reduce the question of the existence of the limit of a function to one concerning the existence of limits of sequences. As we will see, this result will be very useful in subsequent proofs, and also in showing that a given function does not have a limit at a point x_0.

Theorem 4.7. *Let $f : D \subset \mathbb{R} \to \mathbb{R}$ be a given function and x_0 be an accumulation point of a given subset D. Then*

1) $\lim\limits_{x \to x_0} f(x) = l$
if and only if
2) For all sequence $(x_n)_{n\in\mathbb{N}}$, $x_n \in D/\{x_0\}$ *and* $\lim\limits_{n\to+\infty} x_n = x_0$ *we obtain* $\lim\limits_{n\to+\infty} f(x_n) = l$, *(*$l$ *finite or not.)*

Proof. Since x_0 is a limit point of D, there exists a sequence $x_n \in D$ with $x_n \neq x_0$ for all $n \in \mathbb{N}$ and $\lim\limits_{n\to+\infty} x_n = x_0$.

Suppose $\lim\limits_{x\to x_0} f(x) = l$. Let x_n be any sequence in D with $x_n \neq x_0$ for all $n \in \mathbb{N}$ and $\lim\limits_{n\to+\infty} x_n = x_0$. Let $\varepsilon > 0$ be given. Since $\lim\limits_{x\to x_0} f(x) = l$, there exists a $\delta > 0$ such that

$$|f(x) - l| < \varepsilon \quad \text{for all} \quad x \in D, 0 < |x - x_0| < \delta. \tag{4.1}$$

Since $\lim\limits_{n\to+\infty} x_n = x_0$, for the above δ, there exists a positive integer n_0 such that $|x_n - x_0| < \delta$, for all $n \geqslant n_0$. Thus if $n \geqslant n_0$, by (4.1), $|f(x_n) - l| < \varepsilon$. Therefore $\lim\limits_{n\to+\infty} f(x_n) = l$.
Conversely, suppose that $\underset{n\to+\infty}{lim} f(x_n) = l$ for every sequence $x_n \in D$ with $x_n \neq x_0$ for all $n \in \mathbb{N}$ and $\lim\limits_{n\to+\infty} x_n = x_0$. Suppose $\lim\limits_{x\to x_0} f(x) \neq l$. Then there exists an $\varepsilon > 0$ such that for every $\delta > 0$, there exists an $x \in D$ with $0 < |x - x_0| < \delta$ and $|f(x) - l| \geqslant \varepsilon$. For each $n \in \mathbb{N}$, take $\delta = \frac{1}{n}$. Then for each n, there exists $x_n \in D$ such that

$$|x_n - x_0| < \frac{1}{n} \quad \textit{and} \quad |f(x_n) - l| \geqslant \varepsilon.$$

Thus $x_n \longrightarrow x_0$, but $f(x_n)$ does not converge to l. This contradiction proves the result. □

Some limit laws We now state some limit laws for functions.

Theorem 4.8. *Let* $f, g : D \subset \mathbb{R} \to \mathbb{R}$ *be real functions, and* x_0 *an accumulation (cluster) point of* D*. Suppose that* $\lim\limits_{x\to x_0} f(x) = l_1$, $\lim\limits_{x\to x_0} g(x) = l_2$, $l_1, l_2 \in \mathbb{R}$.

Then

1) $\lim\limits_{x\to x_0} (f(x) \pm g(x)) = l_1 + l_2$
2) $\lim\limits_{x\to x_0} (\lambda f(x)) = \lambda l_1$ $(\lambda \in \mathbb{R})$
3) $\lim\limits_{x\to x_0} (f(x) g(x)) = l_1 l_2$
4) $\lim\limits_{x\to x_0} |f(x)| = |l|$
5) $\lim\limits_{x\to x_0} \dfrac{f(x)}{g(x)} = \dfrac{l_1}{l_2}$, *if* $l_2 \neq 0$.

Proof. The proofs are left as an exercises. (To prove the results, use the sequential criterion for limits and the limits laws for sequences). □

Theorem 4.9. *Let $f, g : D \subset \mathbb{R} \to \mathbb{R}$ be functions such that $\forall x \in D : f(x) \leqslant g(x)$. Suppose that $\lim\limits_{x \to x_0} f(x) = l_1$, and $\lim\limits_{x \to x_0} g(x) = l_2$. then $l_1 \leqslant l_2$.*

Proof. Set $F(x) = g(x) - f(x)$ and $L = l_2 - l_1$. It is sufficient to prove that $L \geqslant 0$. We prove the contrapositive. Suppose then that $L < 0$. Let $\varepsilon > 0$ be such that $L + \varepsilon < 0$. Then since $\lim\limits_{x \to x_0} (g - f)(x) = L$, there exists $\delta > 0$ such that if $0 < |x - x_0| < \delta$ then $(g - f)(x) = L + \varepsilon < 0$. Hence, $(g - f)(x) < 0$ for some $x \in D$. We can give another proof using the sequential criterion for limits. $\square$

Corollary 4.2. *$f : D \subset \mathbb{R} \to \mathbb{R}$ be a function and let x_0 be an accumulation point of D. Suppose that $M_1 \leqslant f(x) \leqslant M_2$ for all $x \in D$ and suppose that $\lim\limits_{x \to x_0} f(x) = L$. Then $M_1 \leqslant L \leqslant M_2$.*

Corollary 4.3. *Let $f : D \subset \mathbb{R} \to \mathbb{R}$ be a function and x_0 be an accumulation point of D. If $\lim\limits_{x \to x_0} f(x) = l \in \mathbb{R}$, then $\exists \delta > 0$ such that for $0 < |x - x_0| < \delta$ the functionf is bounded.*

Theorem 4.10. *Let $f, g : D \subset \mathbb{R} \to \mathbb{R}$ be a functions and x_0 be an accumulation point of D. If g is bounded on D and $\lim\limits_{x \to x_0} f(x) = 0$, then $\lim\limits_{x \to x_0} f(x) g(x) = 0$.*

Proof. Use squeeze technique. $\square$

Theorem 4.11. *Suppose that the limit $\lim\limits_{x \to x_0} f(x) = l$, exists. Then $\lim\limits_{x \to x_0} |f(x)| = |l|$.*

Since maxima and minima can be expressed in terms of absolute values, there is a corollary that is sometimes useful.

Corollary 4.4. *Suppose that the limits $\lim\limits_{x \to x_0} f(x) = L$, and $\lim\limits_{x \to x_0} g(x) = M$, exist and that x_0 is a point of accumulation of $D_f \cap D_g$. Then $\lim\limits_{x \to x_0} max\{f(x), g(x)\} = max\{L, M\}$, and $\lim\limits_{x \to x_0} min\{f(x), g(x)\} = min\{L, M\}$.*

Proof. The first follows from the identity $max\{f(x), g(x)\} = \frac{f(x)+g(x)}{2} + \frac{|f(x)-g(x)|}{2}$ and the theorem on limits of sums and the theorem on limits of absolute values. In the same way the second assertion follows from $min\{f(x), g(x)\} = \frac{f(x)+g(x)}{2} - \frac{|f(x)-g(x)|}{2}$. $\square$

4.3 Continuous functions

Let f be a function defined on an interval $[a, b]$. We shall now consider the behaviour of f at points of $[a, b]$.

Continuity at a Point

Definition 4.19. *A function f is said to be continuous at a point $x_0, a < x_0 < b$, if*

$$\lim_{x \to x_0} f(x) = f(x_0)$$

In other words, the function is continuous at x_0, if for each $\varepsilon > 0, \exists \delta > 0$, such that

$$|f(x) - f(x_0)| < \varepsilon, \text{ when } |x - x_0| < \delta$$

Definition 4.20. *A function f is said to be continuous from the left at x_0, if*

$$\lim_{x \to x_0 - 0} f(x) = f(x_0)$$

Also f is continuous from the right at x_0, if

$$\lim_{x \to x_0 + 0} f(x) = f(x_0)$$

Clearly a function is continuous at x_0 if and only if it is continuous from the left as well as from the right.

Definition 4.21. *A function f defined on a closed interval $[a, b]$ is said to be continuous at the end point a if it is continuous from the right at a,*

i.e.,

$$\lim_{x \to a + 0} f(x) = f(a)$$

Also the function is continuous at the end point b of $[a, b]$ if

$$\lim_{x \to b - 0} f(x) = f(b)$$

Thus a function f is continuous at a point x_0 if
(i) $\lim_{x \to x_0} f(x)$ exists, and
(ii) limit equals the value of the function at $x = x_0$.

Continuity in an Interval

A function f is said to be continuous in an interval $[a, b]$ if it is continuous at every point of the interval.

Discontinuous Functions

A function is said to be discontinuous at a point x_0 of its domain if it is not continuous there. The point x_0 is then called a point of discontinuity of the function.

4.4 Types of discontinuities

(i) A function f is said to have a removable discontinuity at $x = x_0$ if $\lim_{x\to x_0} f(x)$ exists but is not equal to the value $f(x_0)$ (which may or may not exist) of the function. Such a discontinuity can be removed by assigning a suitable value to the function at $x = x_0$.

(ii) f is said to have a discontinuity of the first kind at $x = x_0$ if $\lim_{x\to x_0-0} f(x)$ and $\lim_{x\to x_0+0} f(x)$ both exist but are not equal.

(iii) f is said to have a discontinuity of the first kind from the left at $x = x_0$ if $\lim_{x\to x_0-0} f(x)$ exists but is not equal to $f(x_0)$

Discontinuity of the first kind from the right is similarly defined.

(iv) f is said to have a discontinuity of the second kind at $x = x_0$ if neither $\lim_{x\to x_0-0} f(x)$ nor $\lim_{x\to x_0+0} f(x)$ exists.

(v) f is said to have a discontinuity of the second kind from the left at $x = x_0$ if $\lim_{x\to x_0-0} f(x)$ does not exist.

Discontinuity of the second kind from the right may be defined similarly.

Theorems on Continuity

Theorem 4.12. *A function f defined on an interval I is continuous at a point $x_0 \in I$ if for every sequence $\{c_n\}$ in I converging to x_0, we have*

$$\lim_{n\to\infty} f(c_n) = f(x_0)$$

Proof. First let us suppose that the function f is continuous at a point $x_0 \in I$, and $\{c_n\}$ is a sequence in I such that $\lim_{n\to\infty} c_n = x_0$.

Since f is continuous at x_0, therefore, for any given $\varepsilon > 0, \exists$ a $\delta > 0$, such that

$$|f(x) - f(x_0)| < \varepsilon, \text{ when } 0 < |x - x_0| < \delta \tag{4.2}$$

Again, since $\lim_{n\to\infty} c_n = x_0$, therefore, $\exists$ a positive integer m, such that

$$|c_n - x_0| < \delta, \forall n \geq m \tag{4.3}$$

From (4.2), putting $x = c_n$, we have

$$\begin{aligned} & |f(c_n) - f(x_0)| < \varepsilon, \text{ when } |c_n - x_0| < \delta \\ \Rightarrow \quad & |f(c_n) - f(x_0)| < \varepsilon, \quad \forall n \geq m \text{ [using 2]} \\ \Rightarrow \quad & \text{the sequence } \{f(c_n)\} \text{ converges to } f(x_0) \end{aligned}$$

or

$$\lim_{n\to\infty} f(c_n) = f(x_0)$$

Let us now suppose that f is not continuous at x_0, we shall now show that though there exists a sequence (c_n) in I converging to x_0 yet the sequence $(f(c_n))$ does not converge to $f(x_0)$.

Since f is not continuous at x_0, therefore, there exists an $\varepsilon > 0$ such that for every $\delta > 0$, there exists an $x \in I$, such that

$$|f(x) - f(x_0)| \geq \varepsilon, \text{ when } |x - x_0| < \delta$$
$$\lim c_n = x_0.$$

Also by taking $\delta = 1/n$, we find that for each positive integer n, there is a $c_n \in I$, such that

$$|f(c_n) - f(x_0)| \geq \varepsilon, \text{ when } |c_n - x_0| < \frac{1}{n}$$

Thus, the sequence $(f(c_n))$ does not converge to $f(x_0)$, while the sequence (c_n) converges to x_0. $\square$

Remark 4.6. *If* $\lim c_n = x_0 \Rightarrow \lim f(c_n) \neq f(x_0)$, *then* f *is not continuous at* x_0.

Theorem 4.13. *If* f, g *be two functions continuous at a point* x_0, *then the functions* $f + g, f - g, fg$ *are also continuous at* x_0 *and if* $g(x_0) \neq 0$, *then* f/g *is also continuous at* x_0.

The proof is left as an exercise (use the sequential criterion for limits and the laws for sequences).

Example 4.1. *Examine the following function for continuity at the origin:*

$$f(x) = \begin{cases} \frac{xe^{1/x}}{1+e^{1/x}} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Now

$$\lim_{x\to 0-} f(x) = \lim_{x\to 0-} \frac{xe^{1/x}}{1 + e^{1/x}} = 0$$

and

$$\lim_{x\to 0+} f(x) = \lim_{x\to 0+} \frac{x}{e^{-1/x}+1} = 0$$

Thus,

$$\lim_{x\to 0-} f(x) = \lim_{x\to 0+} f(x) = \lim_{x\to 0} f(x) = 0$$

Also

$$\lim_{x\to 0} f(x) = f(0)$$

Thus, the function is continuous at the origin.

Example 4.2. *Show that the function defined as:*

$$f(x) = \begin{cases} \frac{\sin 2x}{x}, & \text{when } x \neq 0 \\ 1, & \text{when } x = 0 \end{cases}$$

has removable discontinuity at the origin.

Solution. Now

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{\sin 2x}{2x} \cdot 2 = 2$$

so that

$$\lim_{x\to 0} f(x) \neq f(0)$$

Hence, the limit exists, but is not equal to the value of the function at the origin. Thus the function has a removable discontinuity at the origin.

Note. The discontinuity can be removed by re-defining the function at the origin such as $f(0) = 2$.

Example 4.3. *Show that the function defined by*

$$f(x) = \begin{cases} x \sin 1/x, & \text{when } x \neq 0 \\ 0, & \text{when } x = 0 \end{cases}$$

is continuous at $x = 0$.

Solution. Now

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \left(x \sin \frac{1}{x} \right) = 0$$

so that

$$\lim_{x\to 0} f(x) = f(0)$$

Hence, f is continuous at $x = 0$.

Example 4.4. *A function f is defined on $[0,1]$ by*

$$f(x) = \begin{cases} -x^2 & \text{if } x \leq 0 \\ 5x-4 & \text{if } 0 < x \leq 1 \\ 4x^2-3x & \text{if } 1 < x < 2 \\ 3x+4 & \text{if } x \geq 2 \end{cases}$$

Examine f for continuity at $x = 0, 1, 2$. Also discuss the kind of discontinuity, if any.

Solution. Now

$$\lim_{x\to 0-} f(x) = \lim_{x\to 0-} \left(-x^2\right) = 0$$
$$\lim_{x\to 0+} f(x) = \lim_{x\to 0+} (5x-4) = -4$$

so that

$$\lim_{x\to 0-} f(x) = f(0) \neq \lim_{x\to 0+} f(x)$$

Thus the function has a discontinuity of the first kind from the right at the origin.

Again

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (5x-4) = 1$$
$$\lim_{x\to 1+} f(x) = \lim_{x\to 1+} \left(4x^2-3x\right) = 1$$

so that

$$\lim_{x\to 1-} f(x) = \lim_{x\to 1+} f(x) = 1 = f(1)$$
$$\lim_{x\to 1} f(x) = f(1)$$

Thus the function is continuous at $x = 1$.

Again

$$\lim_{x\to 2-} f(x) = \lim_{x\to 2-} \left(4x^2-3x\right) = 10$$
$$\lim_{x\to 2+} f(x) = \lim_{x\to 2+} (3x+4) = 10$$

Also $f(2) = 10$

$\Rightarrow$

$$\lim_{x\to 2} f(x) = f(2)$$

Thus, the function is continuous at $x = 2$.

Example 4.5. *Is the function f, where $f(x) = \frac{x-|x|}{x}$ continuous?*

Solution. *For $x < 0, f(x) = \frac{x+x}{x} = 2$, continuous.*

For $x > 0, f(x) = \frac{x-x}{x} = 0$, continuous. The function is not defined at $x = 0$.

Thus, $f(x)$ is continuous for all x except zero.

Example 4.6. *Discuss the kind of discontinuity, if any of the function defined as follows:*

$$f(x) = \begin{cases} \frac{x-|x|}{x}, & \text{when } x \neq 0 \\ 2, & \text{when } x = 0 \end{cases}$$

Solution. *The function is continuous at all points except possibly the origin.*

Let us test at $x = 0$.

Now

$$\lim_{x\to 0-} f(x) = \lim_{x\to 0-} \frac{x+x}{x} = 2$$
$$\lim_{x\to 0+} f(x) = \lim_{x\to 0+} \frac{x-x}{x} = 0$$

and

$$f(0) = 2$$

Thus the function has discontinuity of the first kind from the right at $x = 0$.

Example 4.7. *If $[x]$ denotes the largest integer $\leq x$, then discuss the continuity at $x = 3$ for the function*

$$f(x) = x - [x], \quad \forall x \geq 0$$

Solution. *Now*

$$\lim_{x\to 3-} f(x) = \lim_{x\to 3-} \{x - [x]\} = 3 - 2 = 1$$
$$\lim_{x\to 3+} f(x) = \lim_{x\to 3+} \{x - [x]\} = 3 - 3 = 0$$

and

$$f(3) = 0$$

Thus the function has a discontinuity of the first kind from the left at $x = 3$.

Note. The function is continuous at all points except the integer value $1, 2, 3, \ldots$

Example 4.8. *Prove that the Dirichlet's function f defined on $\mathbf{R}$ by*

$$f(x) = \begin{cases} 1, & \text{when } x \text{ is irrational} \\ -1, & \text{when } x \text{ is rational} \end{cases}$$

is discontinuous at every point.

Solution. First, let a be any rational number so that $f(a) = -1$.

Since in any interval there lie an infinite number of rational and irrational numbers, therefore, for each positive integer n, we can choose an irrational number a_n, such that $|a_n - a| < \frac{1}{n}$.

Thus, the sequence $\{a_n\}$ converges to a. But $f(a_n) = 1$ for all n, and $f(a) = -1$, so that

$$\lim_{n\to\infty} f(a_n) \neq f(a)$$

Thus, the function is discontinuous at any rational number a.

Hence the function is discontinuous at all rational points.

Next, let b be any irrational number. For each positive integer n we can choose a rational number b_n, such that $|b_n - b| < \frac{1}{n}$. Thus, the sequence (b_n) converges to b.

But $f(b_n) = -1$ for all n and $f(b) = 1$.

Therefore $\lim_{n\to\infty} f(b_n) \neq f(b)$.

Hence, the function is discontinuous at all irrational points.

Example 4.9. *Show that the function $f(x)$ defined on $\mathbb{R}$ by*

$$f(x) = \begin{cases} x, & \text{when } x \text{ is irrational} \\ -x, & \text{when } x \text{ is rational} \end{cases}$$

is continuous only at $x = 0$.

Solution. First, let $a \neq 0$ be any rational number, so that $f(a) = -a$. Since in every interval there lie an infinite number of rational and irrational numbers, therefore, for each positive integer n, we can choose an irrational number a_n such that

Thus the sequence (a_n) converges to a.

$$|a_n - a| < \frac{1}{n}$$

But

$$\lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} a_n = a$$

Thus

$$\lim_{n\to\infty} f(a_n) \neq f(a), a \neq 0$$

so that, the function is discontinuous at any rational number, other than zero.

In a similar way the function may be shown to be discontinuous at every irrational point.

It may be seen from above, that the function is continuous at $x = 0$ (i.e., $a = 0$). However, it can be shown to be continuous at $x = 0$ as follows:

Let $\varepsilon > 0$ be given and let $\delta = \varepsilon$ (or any $\delta < \varepsilon$), then

$$|x| < \delta \Rightarrow |f(x) - f(0)| = |-x| = |x| < \varepsilon, \text{ when } x \text{ is rational and}$$
$$|x| < \delta \Rightarrow |f(x) - f(0)| = |x| < \varepsilon, \text{ when } x \text{ is irrational.}$$

Thus

$$|x| < \delta \Rightarrow |f(x) - f(0)| < \varepsilon$$

or

$$\lim_{x\to 0} f(x) = f(0)$$

Hence, the function is continuous at $x = 0$.

4.4.1 Continuous functions on closed intervals

We shall now study some properties of functions which are continuous on closed intervals. In fact, we shall show that a function which is continuous on a closed interval, is bounded, attains its bounds and assumes every value between the bounds.

Theorem 4.14. *If a function is continuous in a closed interval, then it is bounded therein.*

Proof. Let f be a function defined and continuous in a closed interval I.

We shall show that if the function f is not bounded, then it fails to be continuous at some point of the closed interval I

Let, if possible, f be not bounded above, so that for each positive integer $n \exists$ a point $x_n \in I$ such that $f(x_n) > n$

Now $\{x_n\}$, being a sequence in the closed interval I, is bounded and has at least one limit point, say ξ.

A closed interval is a closed set and so $\xi \in I$.

Further, since ξ is a limit point of the sequence (x_n), therefore, there exists (Bolzano-Weirestrass theorem) a subsequence (x_{n_k}) of (x_n) such that $x_{n_k} \to \xi$ as $k \to \infty$.

Also since $f(x_{n_k}) > n_k$, for all k, therefore the sequence $(f(x_{n_k}))$ diverges to ∞.

Thus, there exists a point ξ of I such that a sequence (x_{n_k}) in I converges to ξ, but

$$\lim_{k\to\infty} f(x_{n_k}) \neq f(\xi)$$

Thus, f is not continuous at ξ, which is a contradiction and hence the function is bounded above. By considering a function $-f$, it can be shown in a similar way that the function f is also bounded below. Hence, the function is bounded. □

Theorem 4.15. *If a function f is continuous on a closed interval $[a,b]$, then it attains its bounds at least once in $[a,b]$.*

Proof. If f is a constant function, then evidently, it attains its bounds at every point of the interval.

Let f be a function which is not a constant.

Since f is continuous on the closed interval $[a,b]$, therefore, it is bounded. Let m and M be the infimum and supremum of f. It is to be shown that $\exists$ points, α, β of $[a,b]$ such that

$$f(\alpha) = m, \quad f(\beta) = M$$

Let us consider the case of the supremum.

Suppose f does not attain the supremum M so that the function does not take the value M for any point $x \in [a,b]$, i.e.,

$$f(x) \neq M, \text{ for any } x \in [a,b]$$

Now consider the function

$$g(x) = \frac{1}{M - f(x)}, \quad \forall x \in [a,b]$$

which is positive for all values of x in $[a,b]$.

Evidently the function g is continuous and so bounded in $[a,b]$.

Let $k(>0)$ be its supremum

$$\begin{aligned} &\tfrac{1}{M-f(x)} \leq k, \forall x \in [a,b] \\ \Rightarrow\quad & f(x) \leq M - \tfrac{1}{k}, \forall x \in [a,b] \end{aligned}$$

which contradicts the hypothesis that M is the supremum of f in $[a,b]$. Hence, our supposition that f does not attain the value M leads to a contradiction and therefore f attains its supremum for at least one value in $[a,b]$.

It may similarly be shown that the function also attains its infimum m.

Hence, the function attains its bounds at least once in $[a,b]$.

$\square$

Note It may be observed from the two preceding theorems, that the function f, continuous on the closed interval $[a,b]$, has the least and the greatest values m and M, i.e., the range set of f is bounded with m and M as its smallest and greatest elements. Thus the range set of f is a subset of $[m,M]$. We shall, in fact, show later that the range set of f is $[m,M]$ itself and that f takes up every value between m and M.

4.4.2 Examples

1. The function $f(x) = \frac{1}{1+|x|}$, for real x, is continuous and bounded and attains its supremum for $x = 0$ but does not attain the infimum.

2. The function $f(x) = -\frac{1}{1+|x|}$, for all $x \in \mathbb{R}$, is continuous and bounded, attains its infimum but not the supremum.

3. The function $f(x) = x$, for all $x \in]0,1[$ is continuous and bounded but attains neither the infimum nor the supremum.

Theorem 4.16. *If a function f is continuous at an interior point c of an interval $[a,b]$ and $f(c) \neq 0$, then $\exists$ a $\delta > 0$ such that $f(x)$ has the same sign as $f(c)$, for every $x \in]c-\delta, c+\delta[$.*

Proof. Since the function f is continuous at an interior point c of $[a,b]$, therefore for any $\varepsilon > 0, \exists$ a $\delta > 0$, such that

$$|f(x) - f(c)| < \varepsilon, \quad \forall x \in]c-\delta, c+\delta[$$

or

$$f(c) - \varepsilon < f(x) < f(c) + \varepsilon, \quad \forall x \in]c-\delta, c+\delta[$$

When $f(c) > 0$, taking ε to be less than $f(c)$, we find that

$$f(x) > 0, \forall x \in]c-\delta, c+\delta[$$

When $f(c) < 0$, taking ε to be less than $-f(c)$, we find that

$$f(x) < 0, \forall x \in]c-\delta, c+\delta[$$

Hence the theorem.

□

Corollary 4.5. *If f is continuous at the end point b of $[a, b]$ and $f(b) \neq 0$, then there exists an interval $]b-\delta, b]$, such that $f(x)$ has the sign of $f(b)$ for all x in $]b-\delta, b]$*

A similar result holds for continuity at a.

Note When c is an interior point of the interval, the theorem may be restated as:

Theorem 4.17. *If a function f is continuous at an interior point c of an interval $[a, b]$ and $f(c) \neq 0$, then there exists a neighbourhood N of c wherein $f(x)$ has the same sign as $f(c)$, for all $x \in N$.*

Theorem 4.18. *(intermediate value theorem) If a function f is continuous on a closed interval $[a, b]$ and $f(a)$ and $f(b)$ are of opposite signs, then there exists at least one point $\alpha \in]a, b[$ such that $f(\alpha) = 0$.*

Proof. Let us suppose that $f(a) > 0$ and $f(b) < 0$.

Let S consist of those points of $[a, b]$ for which $f(x)$ is positive, i.e.,

$$S = \{x : a \leq x \leq b \wedge f(x) > 0\}$$

Now

$$f(a) > 0 \Rightarrow a \in S \Rightarrow S \neq \phi$$

Also S is bounded by a and b.

By the order completeness property, S has the supremum, say α, where $a \leq \alpha \leq b$. We shall now show that

(i) $\alpha \neq a, \alpha \neq b$, and

(ii) $f(\alpha) = 0$

(i) First we show that $\alpha \neq a$

Since $f(a) > 0$, therefore $\exists$ a $\delta > 0$ such that

$$f(x) > 0, \quad \forall x \in [a, a+\delta[$$

$\Rightarrow \quad [a, a+\delta[\subseteq S$

$\Rightarrow$ the supremum α of S is greater than or equal to $a+\delta$

$\Rightarrow \alpha \neq a$

Now we shall show that $\alpha \neq b$.

Since $f(b) < 0$, therefore $\exists$ a $\delta > 0$ such that

$$f(x) < 0, \quad \forall x \in]b-\delta, b]$$

$\Rightarrow b-\delta$ is an upper bound of S

$\Rightarrow \alpha \leq b-\delta \Rightarrow \alpha \neq b$

(ii) We shall now show that $f(\alpha) \not> 0$ and $f(\alpha) \not< 0$.

If $f(\alpha) > 0$, then $\exists$ a $\delta > 0$ such that

$$\begin{aligned} f(x) > 0, \quad \forall x \in]\alpha-\delta, \alpha+\delta[\\ \Rightarrow \quad]\alpha-\delta, \alpha+\delta[\subseteq S \end{aligned}$$

Let us choose a positive $\delta_2 < \delta$ such that $\alpha+\delta_2 \in]\, \alpha-\delta, \alpha+\delta\, [\Rightarrow$ a member $\alpha+\delta_2$ of S is greater than the supremum α of S, which is a contradiction.

Therefore

$$f(\alpha) \not> 0$$

Let now $f(\alpha) < 0$, so that $\exists$ a $\delta_1 > 0$ such that

$$f(x) < 0, \quad \forall x \in]\alpha-\delta_1, \alpha+\delta_1[\tag{4.4}$$

Again, since α is the supremum of S, therefore, there exists a member β of S, where $\alpha-\delta_1 < \beta \leq \alpha$ such that

$$f(\beta) > 0$$

But from (4.4), $f(\beta) < 0$, which is a contradiction.

Therefore $\quad f(\alpha) \not< 0$

Thus it follows that $f(\alpha) = 0$.

□

Theorem 4.19. *If a function f is continuous on $[a, b]$ and $f(a) \neq f(b)$, then it assumes every value between $f(a)$ and $f(b)$*

Proof. Let A be any number between $f(a)$ and $f(b)$. We shall show that there exists a number $c \in]a, b[$ such that $f(c) = A$. Consider a function ϕ defined on $[a, b]$ such that

$$\phi(x) = f(x) - A$$

Clearly $\phi(x)$ is continuous on $[a, b]$.

Also

$$\phi(a) = f(a) - A, \text{ and } \phi(b) = f(b) - A$$

so that $\phi(a)$ and $\phi(b)$ are of opposite signs.

Thus the function ϕ is continuous on $[a, b]$ and $\phi(a)$ and $\phi(b)$ are of opposite signs; therefore, by the previous theorem, $\exists c \in]a, b[$ such that

$$\phi(c) = 0 \Longrightarrow f(c) - A \quad = 0 \Rightarrow f(c) = A,$$

□

Corollary 4.6. *A function f, which is continuous on a closed interval $[a, b]$, assumes every value between its bounds.*

Proof. Since the function f is continuous on the closed interval $[a, b]$, therefore, it is bounded and attains its bounds on $[a, b]$, i.e., $\exists$ two numbers α, β in $[a, b]$ such that

$$f(\alpha) = M \text{ and } f(\beta) = m,$$

where M and m are, respectively, the supremum and the infimum of f.

Since f is continuous on $[a, b]$, therefore, it is continuous on $[\beta, \alpha]$ or $[\alpha, \beta]$ as the case may be, and consequently assumes every value between $f(\alpha)$ and $f(\beta)$.

Thus the function assumes every value between its bounds.

We may sum up in other words:

The range of a continuous function whose domain is a closed interval is as well a closed interval. Or, in still better words:

The image of a closed interval under a continuous function (mapping) is a closed interval.

□

4.4.3 Uniform continuity

Let f be a function defined on an interval I. Then by definition, the function is continuous at any point $x_0 \in I$ if for any $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|f(x) - f(x_0)| < \varepsilon, \text{ when } |x - x_0| < \delta.$$

For continuity at any other point $d \in I$, for the same ε, a $\delta_1 > 0$ would exist (not necessarily equal to δ). There is in fact a δ corresponding to each point of I. The number δ in general depends on the selection of ε and the point x_0. However, if a δ could be found which depends only on ε and not on the selection of the point x_0, such a δ would work for the whole interval I on which f is continuous. In such a case, f is said to be uniformly continuous on I. Thus, the notion of uniform continuity is global in character in as much as we talk of uniform continuity only on an interval.

The notion of continuity is, however, local in character in as much as we can talk of continuity at a point.

It may seem to a beginner that the infimum of the set consisting of δ 's corresponding to different points of I would work for the whole of I. But the infimum may be zero. In general, therefore, a δ which may work for the entire interval may not exist, so that every continuous function may not be uniformly continuous.

Definition 4.22. *A function f defined on an interval I is said to be uniformly continuous on I if to each $\varepsilon > 0$ there exists a $\delta > 0$ such that*

$$|f(x_2) - f(x_1)| < \varepsilon, \text{ for arbitrary points } x_1, x_2 \text{ of } I$$
$$\text{for which } |x_1 - x_2| < \delta$$

Theorem 4.20. *A function which is uniformly continuous on an interval is continuous on that interval.*

Let a function f be uniformly continuous on an interval I, so that for a given $\varepsilon > 0$, there corresponds a $\delta > 0$ such that

$|f(x_1) - f(x_2)| < \varepsilon$, where x_1, x_2 are any two points of I for which

$$|x_1 - x_2| < \delta$$

Let $x \in I$, then on taking $x_1 = x$, we find that for $\varepsilon > 0, \exists \delta > 0$ such that

$$|f(x) - f(x_2)| < \varepsilon, \text{ when } |x - x_2| < \delta.$$

Hence the function is continuous at every point $x_2 \in I$, i.e., the function f is continuous on I.

Theorem 4.21. *A function which is continuous on a closed interval is also uniformly continuous on that interval.*

Proof. Let a function f be continuous on a closed interval I. Let, if possible, f be not uniformly continuous on I. Then there exists an $\varepsilon > 0$ such that for any $\delta > 0$, there are numbers $x, y \in I$ for which

$$|f(x) - f(y)| \not< \varepsilon, \text{ when } |x - y| < \delta$$

In particular for each positive integer n, we can find real numbers x_n, y_n in I such that

$$|f(x_n) - f(y_n)| \not< \varepsilon, \text{ when } |x_n - y_n| < 1/n$$

Now (x_n) and (y_n) being sequences in the closed interval I, they are bounded and so each has at least one limit point, say ξ and η respectively.

As a closed interval is a closed set,

therefore

$$\xi \in I, \eta \in I$$

Further since ξ is a limit point of (x_n), there exists a convergent subsequence (x_{n_k}) of (x_n), such that $x_{n_k} \to \xi$.

Similarly, there exists a convergent subsequence (y_{n_k}) of (y_n) such that $y_{n_k} \to \eta$.

Again from above, we find that

$$|f(x_{n_k}) - f(y_{n_k})| \not< \varepsilon, \text{ when } |x_{n_k} - y_{n_k}| < 1/n_k \leq 1/k$$

The second inequality shows that

$$\lim_{k\to\infty} x_{n_k} = \lim_{k\to\infty} y_{n_k}$$
$$\xi = \eta$$

From the first inequality we find that in case the sequences $(f(x_{n_k}))$ and $(f(y_{n_k}))$ converge, the limits to which they converge are different.

We thus have two sequences (x_{n_k}) and (y_{n_k}) both of which converge to ξ but $(f(x_{n_k}))$ and $(f(y_{n_k}))$ do not converge to the same limit.

So f is not continuous at ξ, for, otherwise, the two sequences $(f(x_{n_k}))$ and $(f(y_{n_k}))$ would converge to the same point $f(\xi)$.

Thus we arrive at a contradiction and so the hypothesis that f is not uniformly continuous on I is false.

Hence f is uniformly continuous on I. $\square$

Theorem 4.22. *If $f, g : (a, b) \longrightarrow \mathbb{R}$ are both uniformly continuous, then $f + g$ and $f - g$ (as functions from (a, b) to $\mathbb{R}$) are also uniformly continuous. $fg : (a, b) \longrightarrow \mathbb{R}$ is uniformly continuous.*

Theorem 4.23. *Let f and g be uniformly continuous on $\mathbb{R}$. Then their composition*

$f \circ g$ is also uniformly continuous on $\mathbb{R}$.

Definition 4.23. *A function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is defined to be Lipschitz if there exists a constant $K > 0$ such that for all $a, b \in \mathbb{R}$, we have $|f(a) - f(b)| \leq K|a - b|$*

Theorem 4.24. *A Lipschitz function is uniformly continuous. A periodic and continuous function is uniformly continuous.*

Example 4.10. *Show that the function $f(x) = 1/x$ is not uniformly continuous on $]0, 1]$.*

Solution. Clearly the function is continuous on $]0, 1]$.

It will be uniformly continuous on the given interval if for a given $\varepsilon > 0, \exists$ a $\delta > 0$, independent of the choice of points x and c in $]0, 1]$, such that

$$\left|\frac{1}{x} - \frac{1}{c}\right| < \varepsilon, \text{ when } |x - c| < \delta$$

or

$$\left|\frac{c - x}{cx}\right| < \varepsilon, \text{ when } c - \delta < x < c + \delta \tag{4.5}$$

If we take $c = \delta$, then the interval $]c - \delta, c + \delta[$ becomes $]0, 2\delta[$. Also condition (4.5) must hold for any x in this interval.

But

$$\frac{\delta - x}{\delta x} \to \infty \text{ as } x \to 0,$$

i.e., if we choose x sufficiently close to zero, then condition (4.5)) is violated.

Hence $1/x$ is not uniformly continuous on $]0, 1]$.

Example 4.11. *Show that the function $f(x) = x^2$ is uniformly continuous on $[-1, 1]$.*

Solution. Let x_1, x_2 be any two points of $[-1, 1]$, then

$$\begin{aligned} |f(x_1) - f(x_2)| &= \left|x_1^2 - x_2^2\right| = |x_1 - x_2| \cdot |x_1 + x_2| < \varepsilon \\ \text{when } |x_1 - x_2| &< \frac{1}{2}\varepsilon = \delta \end{aligned}$$

(where δ is independent of the choice of x_1, x_2).

Thus for any $\varepsilon > 0, \exists$ a $\delta = \frac{1}{2}\varepsilon$ such that for any choice of x_1, x_2 in $[-1, 1]$, we have

$$|f(x_1) - f(x_2)| < \varepsilon, \text{ when } |x_1 - x_2| < \frac{1}{2}\varepsilon = \delta$$

Thus the function f is uniformly continuous on $[-1, 1]$.

Inverse functions

Lemma 4.25. *Let A, B be two subset of $\mathbb{R}$. Let $f : A \longrightarrow B$ be a bejective and strictly monotonic function. The f^{-1} is strictly monotonic function, the same monotonic as f.*

Proof. WLOG, we can suppose that f is strictly increasing. let y_1, y_2 be two elements of B such that $y_1 < y_2$, then we prove that $f^{-1}(y_1) < f^{-1}(y_2)$. Since f is bejective then there exist x_1, x_2 such that $f(x_1) = y_1$and $\quad f(x_2) = y_2$. Let us proceed by contrapositive, we suppose that $x_1 \geq x_2$ then $y_1 = f(x_1) \geq f(x_2)y_2$ which is a contradiction with $y_1 < y_2$. □

Lemma 4.26. *let I be an interval of $\mathbb{R}$ and $f : I \Longrightarrow \mathbb{R}$ be a monotonic function such that $f(I)$ be an interval, then f is necessary continuous on I.*

Proof. WLOG, we can suppose that I is not a trivial interval $I = \emptyset, \quad I = a$ and we suppose that f is strictly increasing. Let $a \in I$, if $a > inf(I)$, we prove that $\lim\limits_{x \to a^-} f(x) = f(a)$ and if $a < sup(I)$, we prove that $\lim\limits_{x \to a^+} f(x) = f(a)$. Indeed let $a \in I$, if $a > inf(I)$, by monotonic limit theorem, there exists a real l such that $\lim\limits_{x \to a^-} f(x) = l$, such that $l < f(a)$. Actually, $l = sup\{f(x), x \in I, x < a\}$, we aim to prove that $l = f(a)$. By contrapositive, suppose that $l < f(a)$, then there exists $l < m < f(a)$, there exists $\alpha \in I$ such that $\alpha < a$. By the hypothesis that f is increasing, we get $f(\alpha) \leq l < m < f(a)$. $f(I)$ is an interval, so there exists $c \in I$ such that $f(c) = m$.

- If $c \geq a$, and f is supposed to be increasing, hence $f(c) \geq f(a) > m$,
- If $c < a$, monotonic limit gives $f(c) < f(a) < m$, which is a contradiction, thus $f(a) = l$.

Similarly, with the right limit. □

Lemma 4.27. *let I be an interval of $\mathbb{R}$ and $f : I \Longrightarrow \mathbb{R}$ be a continuous and injective function. Then f is strictly monotonic.*

Proof. By contrapositive principal, we suppose that f isn't strictly monotone, thus

- $\exists x, y \in I : x < y$ and $f(x) \geq f(y)$
- $\exists x', y' \in I : x' > y'$ and $f(x') \leq f(y')$

The segments $[x, x'], [y, y']$ are defined by $[x, x'] = \{tx + (1-t)x', t \in [0,]\}$ and $[y, y'] = \{ty + (1-t)y', t \in [0, 1]\}$. Then let us define the following functions:

$$\alpha : [0, 1] \longrightarrow \mathbb{R} : t \to tx + (1-t)x', \quad \beta : [0, 1] \longrightarrow \mathbb{R} : t \to ty + (1-t)y'.$$

$\alpha(t) and \beta(t)$ belong to the interval I. Now we consider the function defined by

$$\phi : [0, 1] \longrightarrow \mathbb{R} : t \to f(\alpha(t)) - f(\beta(t)).$$

(1) α, β, f are continuous so ϕ,
(2) $\phi(0) = f(x) - f(y) \geq 0$ and $\phi(1) = f(x') - f(y') \leq 0$. The mean theorem value implies that there exists $t_0 \in]0, 1[$ such that $\phi(t_0) = 0$, which means that $f(\alpha(t_0) = f(\beta(t_0)$. Contradiction (f is injective). □

The inverse function theorem for strictly monotonic function

Theorem 4.28. *let I be an interval of $\mathbb{R}$ and $f : I \longrightarrow \mathbb{R}$ be a function. Set $J = f(I)$. Then two of the following properties implie the third one.*

1- J is an interval and $f : I \longrightarrow J$ is a bejection function.

2- f is strictly monotonic on I.

3- f is continuous on I.

more; if 1, 2 and 3 are satisfied, then the inverse function $f^{-1} : J \longrightarrow I$ is continuous, strictly monotonic, the same as f.

Proof.

- If 1 and 2 are satisfied then f is continuous (Lemma (4.26)).
- If 1 and 3 are satisfied then f is strictly monotonic (Lemma (4.27)).
- If 2 and 3 are satisfied then J is an interval (MTV theorem). f is strictly monotonic, thus f is injective and by the way bejective.

If 1, 2 and 3 are satisfied the by (Lemma (4.25)), $f^{-1} : J \longrightarrow I$ is strictly monotonic, the same as f. $f^{-1} : J \longrightarrow I$ realize a bejection from J on I, so f^{-1} satisfies 1 and 2, hence f^{-1} is continuous. □

The fact that the domain of f must be an interval is a necessary condition, see for example the following.

Example 4.12. *Let $g : [0, 1] \cup]2, 3] \longrightarrow [0, 2]$ defined by*

$$g(x) = \begin{cases} x, \ \textit{if } 0 \leq x \leq 1 \\ x - 1, \ \textit{if } 2 < x \leq 3 \end{cases}$$

The inverse is

$$g^{-1}(x) = \begin{cases} y, \ if\ 0 \leq y \leq 1 \\ y+1, \ if\ 1 < y \leq 2 \end{cases}$$

and it is not continuous because of a jump at $y = 1$.

Theorem 4.29. *Let* f *be a continuous and increasing function on the interval* $[a, b]$. *Let* $\alpha = f(a)$, *and* $\beta = f(b)$. *Then*

(1) The image of $[a, b]$ *by* f *is equal to the interval* $[\alpha, \beta]$ *(*$f([a, b]) = [\alpha, \beta]$*)*

(2) There exists an inverse function $x = g(y)$ *of* f *continuous and increasing on* $[\alpha, \beta]$.

Remark 4.7. *The inverse of a decreasing and continuous function* f *on* $[a, b]$ *is a decreasing and continuous function on* $[\alpha, \beta]$, *where* $\alpha = f(a)$, $\beta = f(b)$. *One can consider the function* $-f$.

CHAPTER

5

DIFFERENTIABLE FUNCTIONS

We begin with the definition of the derivative of a function.

Definition 5.1. *Let $I \subset \mathbb{R}$ be an interval and let $x_0 \in I$. We say that $f : I \to \mathbb{R}$ is differentiable at x_0 or has a derivative at x_0 if*

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists. We say that f is differentiable on I if f is differentiable at every point in I.

By definition, f has a derivative at x_0 if there exists a number $L \in \mathbb{R}$ such that for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $|x - x_0| < \delta$ then

$$\left| \frac{f(x) - f(x_0)}{x - x_0} - L \right| < \varepsilon.$$

Derivative function

If f is differentiable at x_0, we will denote $\lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0}$ by $f'(x_0)$, that is,

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}.$$

The rule that sends x_0 to the number $f'(x_0)$ defines a function on a possibly smaller subset $J \subset I$. The function $f' : J \to \mathbb{R}$ is called the derivative of f.

Example 5.1. *Let $f(x) = 1/x$ for $x \in (0, \infty)$. Prove that $f'(x) = -\frac{1}{x^2}$*

Example 5.2. *Let $f(x) = \sin(x)$ for $x \in \mathbb{R}$. Prove that $f'(x) = \cos(x)$.*

Solution

Recall that

$$\sin(x) - \sin(x_0) = 2\sin\left(\frac{x - x_0}{2}\right)\cos\left(\frac{x + x_0}{2}\right)$$

and that $\lim_{x\to 0}\frac{\sin(x)}{x} = 1$. Therefore,

$$\begin{aligned}
\lim_{x\to x_0}\frac{\sin(x) - \sin(x_0)}{x - x_0} &= \lim_{x\to x_0}\frac{2\sin\left(\frac{x-x_0}{2}\right)\cos\left(\frac{x+x_0}{2}\right)}{x - x_0} \\
&= \lim_{x\to x_0}\left(\frac{\sin\left(\frac{x-x_0}{2}\right)}{\frac{x-x_0}{2}}\right)\cos\left(\frac{x + x_0}{2}\right) \\
&= 1\cdot\cos(x_0) = \cos(x_0).
\end{aligned}$$

Hence $f'(x_0) = \cos(x_0)$ for all x_0 and thus $f'(x) = \cos(x)$.

Example 5.3. *Prove by definition that $f(x) = \frac{x}{1+x^2}$ is differentiable on $\mathbb{R}$.*

Solution

$$\begin{aligned}
\frac{f(x) - f(x_0)}{x - x_0} &= \frac{\frac{x}{1+x^2} - \frac{x_0}{1+x_0^2}}{x - x_0} \\
&= \frac{x\left(1 + x_0^2\right) - x_0\left(1 + x^2\right)}{\left(1 + x^2\right)\left(1 + x_0^2\right)(x - x_0)} \\
&= \frac{1 - x_0x}{\left(1 + x_0^2\right)\left(1 + x^2\right)}.
\end{aligned}$$

Now

$$\lim_{x\to x_0}\frac{f(x) - f(x_0)}{x - x_0} = \frac{1 - x_0^2}{\left(1 + x_0^2\right)^2}.$$

Hence, $f'(x_0)$ exists for all $x_0 \in \mathbb{R}$ and the derivative function of f is

$$f'(x) = \frac{1 - x^2}{\left(1 + x^2\right)^2}.$$

Example 5.4. *Prove that $f'(x) = \alpha$ if $f(x) = \alpha x + b$.*

Solution

We have that $f(x) - f(x_0) = \alpha x - \alpha x_0 = \alpha(x - x_0)$. Therefore, $\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0} = \alpha$. This proves that $f'(x) = \alpha$ for all $x \in \mathbb{R}$.

Example 5.5. *Compute the derivative function of $f(x) = |x|$ for $x \in \mathbb{R}$.*

Solution If $x > 0$ then $f(x) = x$ and thus $f'(x) = 1$ for $x > 0$. If $x < 0$ then $f(x) = -x$ and therefore $f'(x) = -1$ for $x < 0$. Now consider $x_0 = 0$. We have that

$$\frac{f(x) - f(x_0)}{x - x_0} = \frac{|x|}{x}.$$

We claim that the limit $\lim_{x \to 0} \frac{|x|}{x}$ does not exist and thus $f'(0)$ does not exist. To see this, consider $x_n = 1/n$. Then $(x_n) \to 0$ and $f(x_n) = 1$ for all n. On the other hand, consider $y_n = -1/n$. Then $(y_n) \to 0$ and $f(y_n) = -1$. Hence, $\lim_{n \to \infty} f(x_n) \neq \lim_{n \to \infty} f(y_n)$, and thus the claim holds by the Sequential criterion for limits. The derivative function f' of f is therefore defined on $A = \mathbb{R} \backslash \{0\}$ and is given by

$$f'(x) = \begin{cases} 1, & x > 0 \\ -1, & x > 0. \end{cases}$$

Hence, even though f is continuous at every point in its domain $\mathbb{R}$, it is not differentiable at every point in its domain. In other words, continuity is not a sufficient condition for differentiability.

Geometric interpretation

Let $x_0 \in \mathbb{R}$ and suppose that the function is differentiable at x_0. Let $T_{(x_0, f(x_0))}$ be the tangent line to the graph of f at the point $(x_0, f(x_0))$, and let (h_n) be a sequence of real numbers decreasing to 0 . Let D_n be the straight line joining the points $(x_0, f(x_0))$ and $(x_0 + h_n, f(x_0 + h_n))$. Therefore for all $n \in \mathbb{N}$, $y = \Delta_n (x - x_0) + f(x_0)$, where $\Delta_i = \frac{f(x_0+h_i) - f(x_0)}{h_i}$, is the equation of D_n. When $n \to +\infty$, the straight lines D_n tend to the tangent line $T_{(x_0, f(x_0))}$ and the slopes $\Delta_n \to f'(x_0)$. We conclude that $y = f'(x_0)(x - x_0) + f(x_0)$ is the equation of the tangent line $T_{(x_0, f(x_0))}$. In other words, $f'(x_0)$ is the slope of the tangent line $T_{(x_0, f(x_0))}$.

Right-hand derivative, left-hand derivative

Let $f : I \longmapsto \mathbb{R}$ be function and suppose that there exists $\delta > 0 :]x_0 - \delta, x_0] \subset I$.

Definition 5.2. *If* $\lim\limits_{x \to x_0^-} \dfrac{f(x) - f(x_0)}{x - x_0}$ *exists (finite), we say that f is left hand differentiable at x_0, or has a left hand derivative at x_0. Noted by $f'(x_0 - 0)\left(f'_-(x_0)\right)$*

Let $f : I \longmapsto \mathbb{R}$ be function and suppose that there exists $\delta > 0 : [x_0, x_0 + \delta, [\subset I$.

Definition 5.3. *IF* $\lim_{x \to x_0^+} \frac{f(x) - f(x_0)}{x - x_0}$ *exists (finite), we say that* f *is right hand differentiable at* x_0*, or has a right hand derivative at* x_0*. Noted by* $f'(x_0 + 0)\left(f'_+(x_0)\right)$.

Remark 5.1. *A function f is differentiable at* x_0 *if and only if both the right-hand derivative and left-hand derivative at* x_0 *exist and both of these derivatives are equal.*

Example 5.6. $f : x \longmapsto |x|$ *defined on* $\mathbb{R}$ *is not differentiable at* 0.

Theorem 5.1. *Suppose that* $f : I \to \mathbb{R}$ *is differentiable at* x_0*. Then* f *is continuous at* x_0.

Proof. To prove that f is continuous at x_0 we must show that $\lim_{x \to x_0} f(x) = f(x_0)$. By assumption $\lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0} = f'(x_0)$ exists, and clearly $\lim_{x \to x_0}(x - x_0) = 0$. Hence we can apply the Limits laws and compute

$$\begin{aligned}
\lim_{x \to x_0} f(x) &= \lim_{x \to x_0} (f(x) - f(x_0) + f(x_0)) \\
&= \lim_{x \to x_0} \left(\frac{f(x) - f(x_0)}{(x - x_0)} (x - x_0) + f(x_0) \right) \\
&= f'(x_0) \cdot 0 + f(x_0) \\
&= f(x_0)
\end{aligned}$$

and the proof is complete. □

Theorem 5.2. *Let* $f : I \to \mathbb{R}$ *and* $g : I \to \mathbb{R}$ *be differentiable at* $x_0 \in I$*. The following hold:*

(i) If $\alpha \in \mathbb{R}$ *then* (αf) *is differentiable and* $(\alpha f)'(x_0) = \alpha f'(x_0)$.

(ii) $(f + g)$ *is differentiable at* x_0 *and* $(f + g)'(x_0) = f'(x_0) + g'(x_0)$.

(iii) fg *is differentiable at* x_0 *and* $(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$.

(iv) If $g(x_0) \neq 0$ *then* (f/g) *is differentiable at* x_0 *and*

$$\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g(x_0)^2}$$

Proof. Parts (i) and (ii) are straightforward. We will prove only (iii) and (iv). For (iii), we have that

$$\begin{aligned}
\frac{f(x)g(x) - f(x_0)g(x_0)}{x - x_0} &= \frac{f(x)g(x) - f(x_0)g(x) + f(x_0)g(x) - f(x_0)g(x_0)}{x - x_0} \\
&= \frac{f(x) - f(x_0)}{x - x_0} g(x) + f(x_0) \frac{g(x) - g(x_0)}{x - x_0}.
\end{aligned}$$

Now $\lim_{x\to x_0} g(x) = g(x_0)$ because g is differentiable at x_0. Therefore,

$$\begin{aligned}\lim_{x\to x_0}\frac{f(x)g(x)-f(x_0)g(x_0)}{x-x_0} &= \lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}g(x)+\lim_{x\to x_0} f(x_0)\frac{g(x)-g(x_0)}{x-x_0}\\ &= f'(x_0)g(x_0)+f(x_0)g'(x_0).\end{aligned}$$

To prove part (iv), since $g(x_0)\neq 0$, then there exist a δ-neighborhood $J=(x_0-\delta, x_0+\delta)$ such that $g(x)\neq 0$ for all $x\in J$. If $x\in J$ then

$$\begin{aligned}\frac{\frac{f(x)}{g(x)}-\frac{f(x_0)}{g(x)}}{x-x_0} &= \frac{f(x)g(x_0)-g(x)f(x_0)}{g(x)g(x_0)(x-x_0)}\\ &= \frac{f(x)g(x_0)-f(x_0)g(x_0)+f(x_0)g(x_0)-g(x)f(x_0)}{g(x)g(x_0)(x-x_0)}\\ &= \frac{\frac{f(x)g(x_0)-f(x_0)g(x_0)}{x-x_0}-\frac{f(x_0)g(x)-f(x_0)g(x_0)}{x-x_0}}{g(x)g(x_0)}\end{aligned}$$

Since $g(x_0)\neq 0$, it follows that

$$\lim_{x\to x_0}\frac{\frac{f(x)}{g(x)}-\frac{f(x_0)}{g(x)}}{x-x_0}=\frac{f'(x_0)g(x_0)-f(x_0)g'(x_0)}{g(x_0)^2}$$

and the proof is complete.

□

We now move to the Chain Rule.

Theorem 5.3. *Let $f: I\to\mathbb{R}$ and $g: J\to\mathbb{R}$ be functions such that $f(I)\subset J$ and let $x_0\in I$. If $f'(x_0)$ exists and $g'(f(x_0))$ exists then $(g\circ f)'(x_0)$ exists and $(g\circ f)'(x_0) = g'(f(x_0))f'(x_0)$.*

Proof. Suppose that there exists a neighborhood of x_0 where $f(x)\neq f(x_0)$. Otherwise, the composite function $(g\circ f)(x)$ is constant in a neighborhood of x_0, and then clearly differentiable at x_0. Consider the function $h: J\to\mathbb{R}$ defined by

$$h(y)=\begin{cases}\frac{g(y)-g(f(x_0))}{y-f(x_0)}, & y\neq f(x_0)\\ g'(f(x_0)), & y=f(x_0).\end{cases}$$

Now

$$\begin{aligned}\lim_{y\to f(x_0)} h(y) &= \lim_{y\to f(x_0)}\frac{g(y)-g(f(x_0))}{y-x_0}\\ &= g'(f(x_0))'\\ &= h(f(x_0)).\end{aligned}$$

Hence, h is differentiable at $f(x_0)$ and therefore h is at $f(x_0)$. Now,

$$\frac{g(f(x)) - g(f(x_0))}{x - x_0} = h(f(x))\frac{f(x) - f(x_0)}{x - x_0}$$

and therefore

$$\begin{aligned}
\lim_{x\to x_0} \frac{g(f(x)) - g(f(x_0))}{x - x_0} &= \lim_{x\to x_0} h(f(x))\frac{f(x) - f(x_0)}{x - x_0} \\
&= h(f(x_0))f'(x_0) \\
&= g'(f(x_0))f'(x_0).
\end{aligned}$$

Therefore, $(g \circ f)'(x_0) = g'(f(x_0))f'(x_0)$ as claimed.

□

Example 5.7. *Compute $f'(x)$ if*

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0. \end{cases}$$

Where is $f'(x)$ continuous?

Solution

When $x \neq 0, f(x)$ is the composition and product of differentiable functions at x, and therefore f is differentiable at $x \neq 0$. For instance, on $A = \mathbb{R}\backslash\{0\}$, the functions $1/x, \sin(x)$ and x^2 are differentiable at every $x \in A$. Hence, if $x \neq 0$ we have that

$$f'(x) = 2x\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right).$$

Consider now $x_0 = 0$. If $f'(0)$ exists it is equal to

$$\begin{aligned}
\lim_{x\to 0} \frac{f(x) - f(x_0)}{x - x_0} &= \lim_{x\to 0} \frac{x^2 \sin\left(\frac{1}{x}\right)}{x} \\
&= \lim_{x\to 0} x\sin\left(\frac{1}{x}\right).
\end{aligned}$$

Using the Squeeze Theorem, we deduce that $f'(0) = 0$. Therefore,

$$f'(x) = \begin{cases} 2x\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0. \end{cases}$$

From the above formula obtained for $f'(x)$, we observe that when $x \neq 0$ f' is continuous since it is the product/difference/composition of continuous functions. To determine continuity of f' at $x = 0$ consider $\lim_{x \to 0} f'(x)$. Consider the sequence $x_n = \frac{1}{n\pi}$, which clearly converges to $x_0 = 0$. Now, $f'(x_n) = \frac{2}{n\pi}\sin(n\pi) - \cos(n\pi)$. Now, $\sin(n\pi) = 0$ for all n and therefore $f'(x_n) = -\cos(n\pi) = (-1)^{n+1}$. The sequence $f'(x_n)$ does not converge and therefore $\lim_{x \to 0} f'(x)$ does not exist. Thus, f' is not continuous at $x = 0$.

Example 5.8. *Compute $f'(x)$ if*

$$f(x) = \begin{cases} x^3 \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Where is $f'(x)$ continuous?

Solution

When $x \neq 0, f(x)$ is the composition and product of differentiable functions, and therefore f is differentiable at $x \neq 0$. For instance, on $A = \mathbb{R} \backslash \{0\}$, the functions $1/x, \sin(x)$ and x^3 are differentiable on A. Hence, if $x \neq 0$ we have that

$$f'(x) = 3x^2 \sin\left(\frac{1}{x}\right) - x\cos\left(\frac{1}{x}\right).$$

Consider now $x_0 = 0$. If $f'(0)$ exists it is equal to

$$\begin{aligned} \lim_{x \to 0} \frac{f(x) - f(x_0)}{x - x_0} &= \lim_{x \to 0} \frac{x^3 \sin\left(\frac{1}{x}\right)}{x} \\ &= \lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) \end{aligned}$$

and using the Squeeze Theorem we deduce that $f'(0) = 0$. Therefore,

$$f'(x) = \begin{cases} 3x^2 \sin\left(\frac{1}{x}\right) - x\cos\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0. \end{cases}$$

When $x \neq 0, f'$ is continuous since it is the product/difference/composition of continuous functions. To determine continuity of f' at $x_0 = 0$ we consider the limit $\lim_{x \to 0} f'(x)$. Now $\lim_{x \to 0} 3x^2 \sin\left(\frac{1}{x}\right) = 0$ using the Squeeze Theorem, and similarly $\lim_{x \to 0} x\cos\left(\frac{1}{x}\right) = 0$ using the Squeeze Theorem. Therefore, $\lim_{x \to 0} f'(x)$ exists and is equal to 0 , which is equal to $f'(0)$. Hence, f' is continuous at $x = 0$, and thus continuous everywhere.

Example 5.9. *Consider the function*

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & x \in \mathbb{Q}\backslash\{0\} \\ x^2 \cos\left(\frac{1}{x}\right), & x \notin \mathbb{Q} \\ 0, & x = 0. \end{cases}$$

Show that $f'(0) = 0$.

5.0.1 Some theorems

Definition 5.4. *Let $f : I \to \mathbb{R}$ be a function and let $x_0 \in I$.*

(i) We say that f has a relative maximum at x_0 if there exists $\delta > 0$ such that $f(x) \leq f(x_0)$ for all $x \in (x_0 - \delta, x_0 + \delta)$.

(ii) We say that f has a relative minimum at x_0 if there exists δ such that $f(x_0) \leq f(x)$ for all $x \in (x_0 - \delta, x_0 + \delta)$.

A point $x_0 \in I$ is called a critical point of $f : I \to \mathbb{R}$ if $f'(x_0) = 0$. The next theorem says that a relative maximum/minimum of a differentiable function can only occur at a critical point.

Theorem 5.4. *(**Pierre Fermat: 1601-1665**) Let $f : I \to \mathbb{R}$ be a function and let x_0 be an interior point of I. Suppose that f has a relative maximum (or minimum) at x_0. If f is differentiable at x_0 then x_0 is a critical point of f, that is, $f'(x_0) = 0$.*

Proof. Suppose that f has a relative maximum at x_0; the relative minimum case is similar. Then for $x \neq x_0$, it holds that $f(x) - f(x_0) \leq 0$ for $x \in (x_0 - \delta, x_0 + \delta)$ and some $\delta > 0$. Consider the function $h : (x_0 - \delta, x_0 + \delta) \to \mathbb{R}$ defined by

$$h(x) = \begin{cases} \frac{f(x)-f(x_0)}{x-x_0}, & x \neq x_0 \\ f'(x_0), & x = x_0 \end{cases}$$

Then the function h is continuous at $x_0 = 0$ because $\lim_{x\to x_0} h(x) = h(x_0)$. Now for $x \in A = (x_0, x_0 + \delta)$ it holds that $h(x) \leq 0$ and therefore $f'(x_0) = \lim_{x\to x_0} h(x) \leq 0$. Similarly, for $x \in B = (x_0 - \delta, x_0)$ it holds that $h(x) \geq 0$ and therefore $0 \leq f'(x_0)$. Thus $f'(x_0) = 0$. □

Corollary 5.1. *If $f : I \to \mathbb{R}$ has a relative maximum (or minimum) at x_0 then either $f'(x_0) = 0$ or $f'(x_0)$ does not exist.*

Example 5.10. *The function $f(x) = |x|$ has a relative minimum at $x = 0$, however, f is not differentiable at $x = 0$.*

Theorem 5.5. ***(Michele Rolle: 1652-1719)*** *Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on $]a, b[$. If $f(a) = f(b)$ then there exists $x_0 \in (a, b)$ such that $f'(x_0) = 0$.*

Proof. Since f is continuous on $[a, b]$ it achieves its maximum and minimum at some point x^* and x_*, respectively, that is $f(x_*) \leq f(x) \leq f(x^*)$ for all $x \in [a, b]$. If f is constant then $f'(x) = 0$ for all $x \in (a, b)$. If f is not constant then $f(x_*) < f(x^*)$. Since $f(a) = f(b)$ it follows that at least one of x_* and x^* is not contained in $\{a, b\}$, and hence there exists $x_0 \in \{x_*, x^*\}$ such that $f'(x_0) = 0$. □

Remark 5.2. *Rolle remains true even when the interval is open, provided that* $\lim\limits_{x \to a^+} f(x) = \lim\limits_{x \to b^-} f(x)$.

We now state and prove the main result of this section.

Theorem 5.6. ***(Mean Value theorem: Lagrange 1736-1813)*** *Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Then there exists $x_0 \in (a, b)$ such that $f'(x_0) = \frac{f(b)-f(a)}{b-a}$.*

Proof. If $f(a) = f(b)$ then the result follows from Rolle's Theorem ($f'(x_0) = 0$ for some $x_0 \in (a, b)$). Let $h : [a, b] \to \mathbb{R}$ be the line from $(a, f(a))$ to $(b, f(b)$, that is,

$$h(x) = f(a) + \frac{f(b) - f(a)}{(b - a)}(x - a)$$

and define the function

$$g(x) = f(x) - h(x)$$

for $x \in [a, b]$. Then $g(a) = f(a) - f(a) = 0$ and $g(b) = f(b) - f(b) = 0$, and thus $g(a) = g(b)$. Clearly, g is continuous on $[a, b]$ and differentiable on (a, b), and it is straightforward to verify that $g'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}$. By Rolle's Theorem, there exists $x_0 \in (a, b)$ such that $g'(x_0) = 0$, and therefore $f'(x_0) = \frac{f(b)-f(a)}{b-a}$.

Theorem 5.7. ***Extended mean value theorem: Cauchy theorem 1789-1857)***

Let f, g be two continuous functions on $[a, b]$ and differentiables on $]a, b[$. If g' does not vanish on $]a, b[$, then there exists a point $x_0 \in]a, b[$ such that $\dfrac{f(b) - f(a)}{g(b) - g(a)} = \dfrac{f'(x_0)}{g'(x_0)}$.

Proof. Consider the function

$$h(x) = (f(x) - f(a))(g(b) - g(a)) - (f(b) - f(a))(g(x) - g(a)).$$

This is continuous on $[a, b]$ and differentiable on (a, b), with

$$h'(x) = f'(x)(g(b) - g(a)) - g'(x)(f(b) - f(a)).$$

Note that $h(a) = 0 = h(b)$. By Rolle's Theorem, there a spot x_0 where $h'(x_0) = 0$. $\square$

Remark 5.3. *Lagrange theorem is a special case of Cauchy theorem:* $g(x) = x, x \in [a, b]$.

Example 5.11. *Prove that* $\forall x > 0 : e^x > 1 + x + \dfrac{x^2}{2}$.

Proof. Cauchy theorem applied with $f(x) = e^u, g(u) = 1 + u + \dfrac{u^2}{2}, u \in [0, x]$. Then

$$\exists x_0 \in]0, x[: \frac{e^x - e^0}{1 + x + \dfrac{x^2}{2} - 1} = \frac{e^{x_0}}{1 + x_0} \text{ but } \frac{e^{x_0}}{1 + x_0} > 1, \forall x_0 > 0.$$

So

$$\frac{e^x - 1}{x + \dfrac{x^2}{2}} > 1 \Rightarrow e^x > 1 + x + \frac{x^2}{2}.$$

$\square$

L'Hospital's Rule And Indeterminate Forms

L'Hospital's Rule (Guillaume L'Hospital (1661-1704)) tells us that if we have an indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, all we need to do is differentiate the numerator and differentiate the denominator and then take the limit.

Theorem 5.8. *Let $f(x)$ and $g(x)$ be continuous functions on an interval containing $x = a$, with $f(a) = g(a) = 0$. Suppose that f and g are differentiable, and that f' and g' are continuous. Finally, suppose that $g'(a) \neq 0$. Then*

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} = \frac{f'(a)}{g'(a)}.$$

Also,

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} = \lim_{x \to a^+} \frac{f'(x)}{g'(x)}$$

and

$$\lim_{x \to a^-} \frac{f(x)}{g(x)} = \lim_{x \to a^-} \frac{f'(x)}{g'(x)}$$

Proof. Since that $f(a) = g(a) = 0$ and $g'(a) \neq 0$. Then, for any $x, f(x) = f(x) - f(a)$ and $g(x) = g(x) - g(a)$. But then,

$$\begin{aligned}
\lim_{x\to a} \frac{f(x)}{g(x)} &= \lim_{x\to a} \frac{f(x)-f(a)}{g(x)-g(a)} \\
&= \lim_{x\to a} \frac{[f(x)-f(a)]/(x-a)}{[g(x)-g(a)]/(x-a)} \\
&= \frac{\lim_{x\to a}([f(x)-f(a)]/(x-a))}{\lim_{x\to a}([g(x)-g(a)]/(x-a))} \\
&= \frac{f'(a)}{g'(a)},
\end{aligned}$$

since, by definition, $f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ and $g'(a) = \lim_{x\to a} \frac{g(x)-g(a)}{x-a}$. Since f' and g' are assumed to be continuous, this is also

$$\frac{\lim_{x\to a} f'(x)}{\lim_{x\to a} g'(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

□

This version is easy to prove, and is good enough to compute limits like

$$\lim_{x\to 0} \frac{\sin(2x)}{x+x^2}.$$

However, it isn't good enough to compute limits like

$$\lim_{x\to 0} \frac{1-\cos(2x)}{x^2}$$

since in that case $g'(0) = 0$. To solve problems like the last one, we need the following version.

Theorem 5.9. *Suppose that f and g are continuous on a closed interval $[a,b]$, and are differentiable on the open interval (a,b). Suppose that $g'(x)$ is never zero on (a,b), and that $\lim_{x\to a^+} f'(x)/g'(x)$ exists, and that $\lim_{x\to a^+} f(x) = \lim_{x\to a^+} g(x) = 0$. Then*

$$\lim_{x\to a^+} \frac{f(x)}{g(x)} = \lim_{x\to a^+} \frac{f'(x)}{g'(x)}.$$

Proof. By assumption, f and g are differentiable to the right of a, and the limits of f and g as $x \to a^+$are zero. Define $f(a)$ to be zero, and likewise define $g(a) = 0$. Since these values agree with the limits, f and g are continuous on some half-open interval $[a,b)$ and differentiable on (a,b).

For any $x \in (a,b)$, we have that f and g are differentiable on (a,x) and continuous on $[a,x]$. By the extended MVT, there is a point c between a and x such that $f'(c)g(x) =$

$f'(x)g(c)$. In other words, $f'(c)/g'(c) = f(x)/g(x)$. Also, as x approaches a, c also approaches a, since c is somewhere between x and a. But then

$$\lim_{x\to a^+}\frac{f(x)}{g(x)} = \lim_{x\to a^+}\frac{f'(c)}{g'(c)} = \lim_{c\to a^+}\frac{f'(c)}{g'(c)}$$

That last expression is the same as $\lim_{x\to a^+} f'(x)/g'(x)$. □

Note that this theorem doesn't require anything about $g'(a)$, just about how g' behaves to the right of a. An analogous theorem applies to the limit as $x \to a^-$(and requires f and g and f' and g' to be defined on an interval that ends at a, rather than one that starts at a). You can combine the two to get a theorem about an overall limit as $x \to a$.

The conclusion of L'Hôpital's Rule relates one limit (of f/g) to another limit (of f'/g'), and not to the value of $f'(a)/g'(a)$. This is what allows the theorem to be used recursively to solve problems.

The inverse is not always true, see the following

Example 5.12. *We have* $\lim\limits_{x\to 0}\dfrac{x^2\sin\frac{1}{x}}{\sin x} = \lim\limits_{x\to 0}\dfrac{x^2\sin\frac{1}{x}}{x} = \lim\limits_{x\to 0} x\sin\dfrac{1}{x} = 0,$
despite that $\lim\limits_{x\to 0}\dfrac{\left(x^2\sin\frac{1}{x}\right)'}{(\sin x)'} = \lim\limits_{x\to 0}\dfrac{2x\sin\frac{1}{x} - \cos\frac{1}{x}}{\cos x}$ *does not exist.*

Remark 5.4. *If* $\dfrac{f'(x)}{g'(x)}$ *is an indeterminate form such* $\frac{0}{0}$ *and we suppose that* $f'(x)$ *and* $g'(x)$ *satisfy L'Hopital theorem hypothesis, then* $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)} = \lim\limits_{x\to a}\dfrac{f'(x)}{g'(x)} = \lim\limits_{x\to a}\dfrac{f''(x)}{g''(x)}$.

Theorem 5.10. *Extended L'Hôpital's Rule Suppose that* f *and* g *are two functions well defined and differentiable on a neighbourhood of a point* a*, may be except at* a *and assume that* $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} g(x) = \infty$*, and that* $g(x) \neq 0$ *et* $g'(x) \neq 0$ *on the candidate neighbourhood. So, if* $\lim\limits_{x\to a}\dfrac{f'(x)}{g'(x)}$ *exists, then* $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)}$ *exists and* $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)} = \lim\limits_{x\to a}\dfrac{f'(x)}{g'(x)}$.

Remark 5.5. *If* $a = \infty$ *the transformation* $x = \dfrac{1}{t}$ *guide us to the case* $a = 0$,

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{t\to 0}\frac{f\left(\frac{1}{t}\right)}{g\left(\frac{1}{t}\right)} = \lim_{t\to 0}\frac{\left(f\left(\frac{1}{t}\right)\right)'}{\left(g\left(\frac{1}{t}\right)\right)'} = \lim_{t\to 0}\frac{-\frac{1}{t^2}f'\left(\frac{1}{t}\right)}{-\frac{1}{t^2}g'\left(\frac{1}{t}\right)} = \lim_{x\to\infty}\frac{f'(x)}{g'(x)}.$$

Theorem 5.11. *Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). If $f'(x) = 0$ for all $x \in (a, b)$ then f is constant on $[a, b]$.*

Proof. Let $y \in (a, b]$. Now f restricted to $[a, y]$ satisfies all the assumptions needed in the Mean Value Theorem. Therefore, there exists $x_0 \in (a, y)$ such that $f'(x_0) = \frac{f(y)-f(a)}{y-a}$. But $f'(x_0) = 0$ and thus $f(y) = f(a)$. This holds for all $y \in (a, b]$ and thus f is constant on $[a, b]$ □

Corollary 5.2. *If $f, g : [a, b] \to \mathbb{R}$ are continuous and differentiable on (a, b) and $f'(x) = g'(x)$ for all $x \in (a, b)$ then $f(x) = g(x) + C$ for some constant C.*

The sign of the derivative f' determines where f is increasing or decreasing.

Theorem 5.12. *Suppose that $f : I \to \mathbb{R}$ is differentiable.*

(i) Then f is increasing if and only if $f'(x) \geq 0$ for all $x \in I$.

(ii) Then f is decreasing if and only if $f'(x) \leq 0$ for all $x \in I$.

Proof. Suppose that f is increasing. Then for all $x, x_0 \in I$ with $x \neq x_0$ it holds that $\frac{f(x)-f(x_0)}{x-x_0} \geq 0$ and therefore $f'(x_0) = \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} \geq 0$. Hence, this proves that $f'(x) \geq 0$ for all $x \in I$.

Now suppose that $f'(x) \geq 0$ for all $x \in I$. Suppose that $x < y$. Then by the Mean Value Theorem, there exists $x_0 \in (x, y)$ such that $f'(x_0) = \frac{f(y)-f(x)}{y-x}$. Therefore, since $f'(x_0) \geq 0$ it follows that $f(y) - f(x) \geq 0$.

Part (ii) is proved similarly. □

Derivative of inverse functions

Theorem 5.13. *Let $f :]a, b[\to \mathbb{R}$, $-\infty \leqslant a < b \leqslant +\infty$ be a function that is both invertible and differentiable. Let $f\left(]a, b[\right) =]c, d[$ $(-\infty \leqslant c < d \leqslant +\infty)$ et $f^{-1} :]c, d[\to]a, b[$, the inverse function of f. If f is differentiable at x_0 such that $f'(x_0) \neq 0$, then f^{-1} is differentiable at $f(x_0)$ and satisfies $\left(f^{-1}\right)'\left(f(x_0)\right) = \dfrac{1}{f'(x_0)}$.*

Proof. Use composition derivative formula. □

5.1 Higher order derivatives

Let $f :]a, b[\longmapsto \mathbb{R}$ be a differentiable function on $]a, b[$. Let $g = f' :]a, b[\longmapsto \mathbb{R}$ be the derivative function by setting $g(x) = f'(x), \forall x \in]a, b[$.

Definition 5.5. *At $x_0 \in]a,b[$ the derivative of the derivative of a function f if it exists, is called the second derivative of that function and is denoted by one of the symbols $f''(x_0), \dfrac{d^2}{dx^2} f(x_0)$.*

If the derivative of $n \in \mathbb{N}$ exists, then, we denote it by $f^n(x_0)$ or $\dfrac{d^n}{dx^n} f(x_0)$ and if for all $x \in]a,b[, f^n(x)$ exists, then the derivative of order $n+1$ is defined by $f^{(n+1)}(x) = \left(f^{(n)}(x)\right)'$, if it exists.

Example 5.13. $\forall n \in \mathbb{N}, \forall x \in \mathbb{R}$,

$$(\sin x)^{(n)} = \sin\left(x + n\frac{\pi}{2}\right), (\cos x)^{(n)} = \cos\left(x + n\frac{\pi}{2}\right)$$

By induction we have
For $n = 1$, *one has* $(\sin x)' = \cos x = \sin\left(x + \dfrac{\pi}{2}\right)$, *is true.*

Suppose that it is true for n.

So $(\sin x)^{(n+1)} = \left(\sin^{(n)} x\right)' \left(\sin\left(x + n\dfrac{\pi}{2}\right)\right)' = \cos\left(x + n\dfrac{\pi}{2}\right) = \sin\left(x + n\dfrac{\pi}{2} + \dfrac{\pi}{2}\right) = \sin\left(x + (n+1)\dfrac{\pi}{2}\right)$

which is true too.

Technical fact

If $y = f(x)$ is a function of time that describes the position of a moving object, then:

1. The first derivative represents the velocity of the object.
2. The second derivative represents the acceleration of the object.
3. The third derivative represents the jerk of the object.

Definition 5.6. *Let $A \subset \mathbb{R}$. Then $f \in C^{(n)}(A)$, $(n \in \mathbb{N}) \iff \forall x \in A$, $f^{(n)}(x)\,(n \in \mathbb{N})$ exists and $f^{(n)} \in C(A)\,(n \in \mathbb{N})$. f is said to be n times continuously differentiable.*

Definition 5.7. $C^{(\infty)}(A) = \bigcap\limits_{n=1}^{\infty} C^{(n)}(A)$

$f \in C^{(\infty)}(A) \iff \forall n \in \mathbb{N} : f \in C^{(n)}(A)$, f *is said to be infinitely differentiable.*

Example 5.14. *Functions* $\sin x, \cos x, e^x, x \in \mathbb{R}$ *belong to* $C^{(\infty)}(\mathbb{R})$.

Theorem 5.14. *(Leibniz formula) Let $\{f,g\} \subset C^{(n)}(]a,b[)$ for $n \in \mathbb{N}$. Then $fg \in C^{(n)}(]a,b[)$, and we have*

$$(fg)^{(n)} = \sum_{k=0}^{n} C_n^k f^{(k)} g^{(n-k)}.$$

Proof. By induction. $\square$

Example 5.15. *Calculate* $\left(x^2 \sin 2x\right)^{(10)}$. *One has* $f(x) = x^2, g(x) = \sin 2x$ *and* $f'(x) = 2x,\ f''(x) = 2,\ f'''(x) = ... = f^{10}(x) = 0$

$$\begin{aligned} g'(x) &= 2\sin\left(2x + \frac{\pi}{2}\right) = 2\cos 2x,\ g''(x) = 2^2 \sin\left(2x + 2\frac{\pi}{2}\right) = 2^2(-\sin 2x), ..., \\ g^{(10)}(x) &= 2^{10}\sin\left(2x + 10\frac{\pi}{2}\right) = 2^{10}\sin(2x + 5\pi) = -2^{10}\sin 2x \end{aligned}$$

So

$$\begin{aligned} \left(x^2 \sin 2x\right)^{(10)} &= f^{(0)}g^{(10)} + C_{10}^1 f' g^{(9)} + C_{10}^2 f'' g^{(8)} + C_{10}^3 f'' g^{(7)} \\ &= -2^{10}x^2 \sin 2x + 10(2x)\, 2^9 \sin\left(2x + 9\frac{\pi}{2}\right) + \frac{10.9}{2}\sin\left(2x + 8\frac{\pi}{2}\right) \\ &= -2^{10}x^2 \sin 2x + 2^{10}10x\cos 2x + 8^8.10.9 \sin 2x \\ &= -2^{10}\left(x^2 \sin 2x - 10x\cos 2x - \frac{45}{2}\sin 2x\right). \end{aligned}$$

Local and absolute extrema

An extremum (or extreme value) of a function is a point at which a maximum or minimum value of the function is obtained in some interval. A local extremum (or relative extremum) of a function is the point at which a maximum or minimum value of the function in some open interval containing the point is obtained.

An absolute extremum (or global extremum) of a function in a given interval is the point at which a maximum or minimum value of the function is obtained. Frequently, the interval given is the function's domain, and the absolute extremum is the point corresponding to the maximum or minimum value of the entire function.

Definition 5.8. *Let* $x_0 \in I$ *and let* $f : I \to \mathbb{R}$ *be a function, where* I *is a real interval.*

We say that f *reaches at* x_0 *a local maximum if there exists* $\eta > 0$ *such that* $[x_0 - \eta, x_0 + \eta] \subset I$ *and for all* $x \in [x_0 - \eta, x_0 + \eta]$ *we have* $f(x) \leq f(x_0)$.

We say that f *reaches at* x_0 *a local minimum if there exists* $\eta > 0$ *such that* $[x_0 - \eta, x_0 + \eta] \subset I$ *and for all* $x \in [x_0 - \eta, x_0 + \eta]$ *we have* $f(x) \geq f(x_0)$. *We say that* f *reaches at* x_0 *a local extremal value if* f *reaches at* x_0 *a local maximum or minimum.*

Theorem 5.15. *If* $f : I \to \mathbb{R}$ *reaches at* $x_0 \in I$ *a local extremal value and* f *is differentiable at* x_0*, then* $f'(x_0) = 0$.

Proof. If f reaches at x_0 a local maximum, then

$$f'(x_0) = f'\left(x_0^+\right) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \leq 0$$

and

$$f'(x_0) = f'\left(x_0^-\right) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \geq 0$$

Hence $f'(x_0) = 0$. $\square$

Remark 5.6. *The reciprocal is false since the function $f(x) = x^3$ has $f'(0) = 0$ but it does not reach an extremal value at 0.*

Theorem 5.16. *Suppose that x_0 is a stationary (critical) point of a given function $f\,(i,e : f'(x) = 0)$ and suppose that the second derivative of f is continuous in some neighbourhood of x_0.*

If $f''(x_0) < 0$, then f admits a local maximum local at x_0;

If $f''(x_0) > 0$, then f admits a local minimum local at x_0.

Theorem 5.17. *Suppose that $f'(x_0) = f''(x_0) = ... = f^{(n)}(x_0) = 0$ and suppose that $f^{(n+1)}(x) \neq 0$ is continuous in some neighbourhood of x_0. If $n+1$ is even and $f^{(n+1)}(x_0) < 0$, then f admits a local maximum at x_0; If $n+1$ is even and $f^{(n+1)}(x_0) > 0$, then f admits a local minimum at x_0; If $n+1$ is odd, then f does not have any local extrema at x_0.*

Example 5.16. *Let be the function $f(x) = e^x + e^{-x} + 2\cos x$, $f'(x) = e^x - e^{-x} - 2\sin x$, remark that $x = 0$ is a stationary point. Then*

$$f''(x) = e^x + e^{-x} - 2\cos x, f''(0) = 0$$

$$f'''(x) = e^x - e^{-x} + 2\sin x, f'''(0) = 0$$

$$f^{(4)}(x) = e^x + e^{-x} + 2\cos x, f^{(4)}(0) = 4 > 0, (n+1) = 4$$

so $x = 0$ is a local minimum point.

The following theorem is stated under weak conditions then the above one.

Theorem 5.18. *Suppose that the function f is continuous on an interval $]x_0 - \delta, x_0 + \delta[$ $(\delta > 0)$ and differentiable on $]x_0 - \delta, x_0[$ and on $]x_0, x_0 + \delta[$. Assume that*

$f''(x) \geqslant 0, (resp., \leqslant 0)$ on $]x_0 - \delta, x_0[$ (3)

$f''(x) \leqslant 0, (resp., \geqslant 0)$ on $]x_0, x_0 + \delta[$ (4)

Then the function f admits a local maximum (resp. local minimum) at x_0. Remark that the existence of $f'(x_0)$ isn't mandatory.

Example 5.17. *Find the extrema points of the function $f(x) = x^{\frac{1}{3}}(1-x)^{\frac{2}{3}}$*

First we calculate derivative: $f'(x) = \dfrac{\frac{1}{3} - x}{\sqrt[3]{x^2(1-x)}} = 0$

we can easily check that $x_1 = \dfrac{1}{3}$ is a stationary point. The derivative at points $x_2 = 0$ et $x_3 = 1$ does not exist. Let $0 < \delta < \dfrac{1}{3}$, then:

$f'\left(\frac{1}{3}-\delta\right)>0,\ f'\left(\frac{1}{3}+\delta\right)<0$
$f'(-\delta)>0,\ f'(\delta)>0$
$f'(1-\delta)<0,\ f'(1+\delta)>0$

So for $x_1=\frac{1}{3}$ *the function admits a maximum, for* $x_2=0$*, there is no extrema. For* $x_3=1$ *the function admits a minimum.*

Bounds of a function

Suppose we have to find the maximum (resp, minimum) of a continuous function on an interval $[a,b]$ three cases only are possible:

1) $x_0=a$ 2) $x_0=b$ 3) $x_0\in\,]a,b[$

If $x_0\in\,]a,b[$, then the function f admits a local extrema at x_0, that is a critical point (either stationary or a point such that the derivative does not exist).

If $\{x_1,...,x_n\}$, is a finite set, then

$$\max_{x\in[a,b]}f(x)=\max\{f(a),f(b),f(x_1),...,f(x_n)\}$$
$$\min_{x\in[a,b]}f(x)=\min\{f(a),f(b),f(x_1),...,f(x_n)\}$$

Example 5.18. *Find local extrema of the function* $f(x)=x^3-3x+3$ *on* $\left[-3,\frac{3}{2}\right]$

We have $f'(x)=3x^2-3=0\Leftrightarrow x_1=-1,x_2=1$

Since $f(-1)=5,\ f(1)=1,\ f(-3)=-15,\ f\left(\frac{3}{2}\right)=\frac{15}{8}$

then $\max\limits_{x\in\left[-3,\frac{3}{2}\right]}f(x)=5$ *et* $\min\limits_{x\in\left[-3,\frac{3}{2}\right]}f(x)=-15.$

Convexity of a curve, point of inflection

There are different types of functions. They are classified according to the categories. One such category is the nature of the graph. Depending upon the nature of the graph, the functions can be divided into two types namely, convex Function and concave Function Both the concavity and convexity can occur in a function once or more than once. The point where the function is neither concave nor convex is known as inflection point or the point of inflection.

Definition 5.9. *If a curve opens in an upward direction or it bends up to make a shape like a cup, it is said to be concave up or convex down. If a curve bends down or resembles a cap, it is known as concave down or convex up. In other words, the tangent lies underneath the curve if the slope of the tangent increases by the increase in an independent variable.*

Remark 5.7. *If* f *is a differentiable function, then when* $f''\geq 0$*, we have a portion of the graph where the gradient is increasing, so the graph is convex at this section. When* $f''\leq 0$*, we have a portion of the graph where the gradient is decreasing, so the graph is concave at this section. .*

Definition 5.10. *The point of inflection or inflection point is a point in which the concavity of the function changes. It means that the function changes from concave down to concave up or vice versa. In other words, the point in which the rate of change of slope from increasing to decreasing manner or vice versa is known as an inflection point. Those points are certainly not local maxima or minima, but they are stationary points too.*

Example 5.19. *Let the curve defined by* $y = 1 + \sqrt[3]{x}$.

Check the concavity (convexity) at points $A\left(-1, 0\right)$, $B\left(1, 2\right)$. *On a* $y''\left(x\right) = -\frac{2}{9}x^{-\frac{5}{3}}$, $y''\left(-1\right) = \frac{2}{9} > 0$, $y''\left(1\right) < 0$, *so at point* A, *the curve is concave up or convex down, and at the point* B, *it is concave down or convex up.*

Corollary 5.3. *If* x_0 *is a point of inflection of a curve* $y = f\left(x\right)$ *and if the second derivative* f'' *exists at* x_0, *then one has necessary* $f''\left(x_0\right) = 0$.

Theorem 5.19. *If* f *is such that the derivative* f''' *is continuous at* x_0 *and* $f''\left(x_0\right) = 0$ *with* $f'''\left(x_0\right) \neq 0$, *then the curve* $y = f\left(x\right)$ *admits a point of inflection at* x_0.

Example 5.20. *Consider* $y = x^3 - 3x^2 - 9x + 9$. *We have* $y' = 3x^2 - 6x - 9$, $y'' = 6x - 6$. *Then* $y''\left(1\right) = 0$ *and* $y'''\left(1\right) = 6 \neq 0$. *So* $x = 1$ *is an inflection point.*

Remark 5.8. *A curve can admit an inflection point at a given point* x_0 *despite that* $f''\left(x_0\right)$ *does not exist.*

Theorem 5.20. *Let* f *be a function such that:* $f''\left(x_0\right) = ... = f^{(n)}\left(x_0\right) = 0$, $f^{(n+1)}$ *is continuous at* x_0 *and* $f^{(n+1)}\left(x_0\right) \neq 0$, *if* n *is odd, then the curve* $y = f\left(x\right)$ *is convex up or convex down according to* $f^{(n+1)}\left(x_0\right) < 0$ *or* $f^{(n+1)}\left(x_0\right) > 0$. *If* n *is even, then* x_0 *is a point of inflection.*

Example 5.21. *Let* $y = x^5$. *we have* $y'\left(x\right) = 5x^4$,

$y'' = 20x^3$, $\quad y''\left(0\right) = 0$
$y'''\left(x\right) = 60x^2$, $\quad y'''\left(0\right) = 0$
$y^{(4)}\left(x\right) = 60.2x$, $\quad y^{(4)}\left(0\right) = 0$
$y^{(5)}\left(x\right) = 120$, $\quad y^{(5)}\left(0\right) = 120 \neq 0$
$n = 4$ *even, then* $x_0 = 0$ *is an inflection point.*

Asymptote of a curve An asymptote is a straight line that constantly approaches a given curve but does not meet at any infinite distance. In other words, Asymptote is a line that a curve approaches as it moves towards infinity. The curves visit these asymptotes but never overtake them.

Definition 5.11. *The line $x = a$ is said to be a vertical asymptote of a continuous curve $y = f(x)$ if at least one the two limits $\lim\limits_{x \longrightarrow a^+} f(x)$, $\lim\limits_{x \longrightarrow a^-} f(x)$ is infinite.*

If a curve $y = f(x)$ is defined for $x > A$ (resp.$x < A$), then $y = ax + b$ is said to be oblique asymptote of the curve $y = f(x)$ for $x \longrightarrow +\infty$(resp.$x \longrightarrow -\infty$) if $f(x) = ax + b + \alpha(x)$ where $\lim\limits_{x \longrightarrow +\infty} \alpha(x) = 0$(resp.$x \longrightarrow +\infty$) (in other words $|f(x) - ax - b|$ is infinitely small with respect to pour $x \longrightarrow +\infty$(resp.$x \longrightarrow -\infty$)).

A horizontal asymptote is a horizontal line, $y = a$, that has the property that either: $\lim\limits_{x \longrightarrow +\infty} f(x) = a$ or $\lim\limits_{x \longrightarrow -\infty} f(x) = a$. This means, that as x approaches positive or negative infinity, the function tends to a constant value a.

Example 5.22. *Consider $y = \dfrac{8}{x-2}$. The line $x = 2$ is a vertical asymptote because $\lim\limits_{x \longrightarrow 2^+} \dfrac{8}{x-2} = +\infty$, $\lim\limits_{x \longrightarrow 2^-} \dfrac{8}{x-2} = -\infty$.*

Example 5.23. *Let $y = \dfrac{x^2 + x}{x - 1} = 0$. Since $f(x) = x + 2 + \dfrac{2}{x-1}$ et $\lim\limits_{x \longrightarrow \infty} \dfrac{2}{x-1} = 0$, then the line $y = x + 2$ is an oblique asymptote of the curve $y = \dfrac{x^2 + x}{x - 1}$ for $x \longrightarrow +\infty$ and for ($x \longrightarrow -\infty$).*

Theorem 5.21. *Let $y = f(x)$ be a given curve. $y = f(x)$ admits an oblique asymptote of the form $y = ax + b$ for $x \longrightarrow +\infty$ ($x \longrightarrow -\infty$) if and only if the following limits*

$$\lim_{\substack{x \longrightarrow +\infty \\ (x \longrightarrow -\infty)}} \frac{f(x)}{x} = a, \lim_{\substack{x \longrightarrow +\infty \\ (x \longrightarrow -\infty)}} [f(x) - ax] = b$$

exist.

Remark 5.9. *The existence of the limits*

$$\lim_{\substack{x \longrightarrow +\infty \\ (x \longrightarrow -\infty)}} \frac{f(x)}{x} = a, \lim_{\substack{x \longrightarrow +\infty \\ (x \longrightarrow -\infty)}} [f(x) - ax] = b$$

is necessary. Indeed for the curve

$y = \sqrt{x}\,(x \geqslant 0)$ we have $\lim\limits_{x \longrightarrow +\infty} \dfrac{\sqrt{x}}{x} = 0 = a$ and $\lim\limits_{x \longrightarrow +\infty} [\sqrt{x} - 0x] = +\infty$, which means $b = +\infty$, so this curve does not have asymptote.

Example 5.24. *Let the curve defined by $y = xe^{\frac{1}{x^2}}$*

1) *Vertical asymptote*

$\lim\limits_{x \longrightarrow 0^+} xe^{\frac{1}{x^2}} = \lim\limits_{y \longrightarrow +\infty} \dfrac{e^{y^2}}{y} = +\infty$, $\lim\limits_{x \longrightarrow 0^-} xe^{\frac{1}{x^2}} = \lim\limits_{y \longrightarrow -\infty} \dfrac{e^{y^2}}{y} = -\infty$.

then $x = 0$ is a vertical asymptote.

2) *oblique asymptote*

$a = \lim_{x \longrightarrow \pm\infty} x \frac{e^{\frac{1}{x^2}}}{x} = \lim_{x \longrightarrow \pm\infty} x e^{\frac{1}{x^2}} = 1,$
$b = \lim_{x \longrightarrow \pm\infty} \left[x e^{\frac{1}{x^2}} - x \right] = \lim_{x \longrightarrow \pm\infty} \left[x \left(1 + \frac{1}{x^2} \right) - x \right] = 0.$
thus $b = 0$. So $y = x$ is an oblique asymptote of the considered curve.

5.2 Convex Functions

5.2.1 Introduction

Convexity is the study of convex sets and functions, it constitutes a branch of geometry and analysis that unites phenomena that are at first sight totally dissimilar. It intervenes at various levels in very varied branches of mathematics (number theory, combinatorial problems, optimization, functional analysis, etc.). In analysis, the property of convexity is relatively sought after, since as we will see in the following, it offers a good number of nice properties on the function.

Definition 5.12. *A function $f : I \longrightarrow \mathbb{R}$ is said to be convex when:* $\forall (x_1, x_2) \in I, \forall \lambda \in [0, 1]$

$$f(\lambda x_1 + (1 - \lambda) x_2) \leqslant \lambda f(x_1) + (1 - \lambda) f(x_2)$$

this means that for all x_1 and x_2 of I, the line segment connecting the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$ are located above the curve representing f.

Definition 5.13. *f will be said to be concave if $-f$ is convex i.e.* $\forall (x_1, x_2) \in I, \forall \lambda \in [0, 1]$

$$f(\lambda x_1 + (1 - \lambda) x_2) \geqslant \lambda f(x_1) + (1 - \lambda) f(x_2)$$

Definition 5.14. *A function $f : I \longrightarrow \mathbb{R}$ is said to be strictly convex if*
$\forall (x_1, x_2) \in I, x_1 \neq x_2, \forall \lambda \in [0, 1]$

$$f(\lambda x_1 + (1 - \lambda) x_2) < \lambda f(x_1) + (1 - \lambda) f(x_2)$$

Definition 5.15. *A function $f : I \longrightarrow \mathbb{R}$ is said to be strictly concave if*
$\forall (x_1, x_2) \in I, x_1 \neq x_2, \forall \lambda \in [0, 1]$

$$f(\lambda x_1 + (1 - \lambda) x_2) > \lambda f(x_1) + (1 - \lambda) f(x_2)$$

Examples:

1. The function $f : x \longrightarrow x^2$ is convex on $\mathbb{R}$

Indeed for everyone x, y of I and for $\lambda \in [0,1]$ we have:

$$\begin{aligned}
[\lambda x + (1-\lambda) y]^2 &\leqslant \lambda x^2 + (1-\lambda) y^2 \\
&\Longleftrightarrow \\
\lambda^2 x^2 + 2\lambda (1-\lambda) xy + (1-\lambda)^2 y^2 &\leqslant \lambda x^2 + (1-\lambda) y^2 \\
&\Longleftrightarrow \\
\lambda^2 \left[-x^2 + 2xy - y^2\right] - \lambda \left[-x^2 + 2xy - y^2\right] &\leqslant 0 \\
&\Longleftrightarrow \\
\lambda (1-\lambda) \left[-x^2 + 2xy - y^2\right] &\leqslant 0 \\
&\Longleftrightarrow \\
-\lambda (1-\lambda) \ (x-y)^2 &\leqslant 0 \text{ which is true}
\end{aligned}$$

2. Affine functions are both convex and concave.

3. La fonction exp est strictement convexe sur $\mathbb{R}$.

4. La fonction log est strictement concave sur $\mathbb{R}^+$.

5.2.2 Properties of convex functions

Proposition 5.1. *Let $f : I \to \mathbb{R}$ be a function on the interval I. The following assertions are equivalent.*

1. *f is convex on I,*

2. *For any collection of positive real numbers $\alpha_1, \alpha_2, \cdots, \alpha_n$ with $\alpha_1 + \alpha_2 + \cdots + \alpha_n = 1$ and any collection of real numbers $x_1, x_2, \cdots, x_n$ in I we have*

$$f(\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n) \leq \alpha_1 f(x_1) + \alpha_2 f(x_2) + \cdots + \alpha_n f(x_n).$$

Proof. The implication **2** $\Rightarrow$ **1** is clear. The implication **1** $\Rightarrow$ **2** is obtained by induction. If $n = 2$, then $\alpha_2 = 1 - \alpha_1$ and the convexity of f leads to

$$f(\alpha_1 x_1 + \alpha_2 x_2) \leq \alpha_1 f(x_1) + \alpha_2 f(x_2)$$

Suppose that it is true at the rank n and let us prove that it is true at the rank $n+1$.

$$\begin{aligned}
& f\left(\alpha_1 x_1+\alpha_2 x_2+\cdots+\alpha_n x_n+\alpha_{n+1} x_{n+1}\right) \\
= & f\left(\left(\sum_{j=1}^{j=n} \alpha_j\right)\left(\sum_{k=1}^{k=n} \frac{\alpha_k}{\left(\sum_{j=1}^{j=n} \alpha_j\right)} x_k\right)+\alpha_{n+1} x_{n+1}\right) \\
\leq & \left(\sum_{j=1}^{j=n} \alpha_j\right) f\left(\sum_{k=1}^{k=n} \frac{\alpha_k}{\left(\sum_{j=1}^{j=n} \alpha_j\right)} x_k\right)+\alpha_{n+1} f\left(x_{n+1}\right) \\
\leq & \left(\sum_{j=1}^{j=n} \alpha_j\right) \sum_{k=1}^{k=n} \frac{\alpha_k}{\left(\sum_{j=1}^{j=n} \alpha_j\right)} f\left(x_k\right)+\alpha_{n+1} f\left(x_{n+1}\right) \\
= & \alpha_1 f\left(x_1\right)+\alpha_2 f\left(x_2\right)+\cdots+\alpha_n f\left(x_n\right)+\alpha_{n+1} f\left(x_{n+1}\right)
\end{aligned}$$

$\square$

Theorem 5.22. *Let $f : I \to \mathbb{R}$ be a function defined on the interval I. The following propositions are equivalents*

1. *f is convex on I,*
2. *For all $x, y, z \in I, x < y < z$ implies $\frac{f(y)-f(x)}{y-x} \leq \frac{f(z)-f(x)}{z-x}$,*
3. *For all $x, y, z \in I, x < y < z$ implies $\frac{f(y)-f(x)}{y-x} \leq \frac{f(z)-f(y)}{z-y}$,*
4. *For all $x, y, z \in I, x < y < z$ implies $\frac{f(z)-f(x)}{z-x} \leq \frac{f(z)-f(y)}{z-y}$.*

Proof. We prove **1** $\Leftrightarrow$ **2**, the other equivalences are checked similarly.

- **1** $\Rightarrow$ **2**. For $x < y < z$ we write $y = \mu x + (1-\mu)z$ where $\mu = \frac{z-y}{z-x} \in [0,1]$. From the convexity of f we deduce

$$f(y) \leq \mu f(x) + (1-\mu) f(z) = \frac{z-y}{z-x} f(x) + \frac{y-x}{z-x} f(z).$$

Adding $-f(x)$ and dividing by $(y-x)$ we get

$$\frac{f(y)-f(x)}{y-x} \leq \frac{f(z)-f(x)}{z-x}$$

- **2** $\Rightarrow$ **1**. Let $x_1, x_2 \in I$ with $x_1 < x_2$ and let $\lambda \in [0,1]$. It is clear that if $\lambda = 0$ or $\lambda = 1$ then $f\left(\lambda x_1 + (1-\lambda) x_2\right) \leq \lambda f\left(x_1\right) + (1-\lambda) f\left(x_2\right)$.

If $\lambda \in (0,1)$ then setting $x = x_1, y = \lambda x_1 + (1-\lambda) x_2$ and $z = x_2$ we get $x < y < z$ and consequently

$$\frac{f(y)-f(x)}{y-x} \leq \frac{f(z)-f(x)}{z-x}$$

Leading to

$$f(y) \leq \frac{z-y}{z-x} f(x) + \frac{y-x}{z-x} f(z),$$

that is

$$f\left(\lambda x_1 + (1-\lambda) x_2\right) \leq \lambda f\left(x_1\right) + (1-\lambda) f\left(x_2\right)$$

□

Definition 5.16. *Given $f : I \longrightarrow \mathbb{R}$ and $\alpha \in I$, the application $\varphi_\alpha : I \setminus \{\alpha\} \longrightarrow \mathbb{R}$ defined by $\varphi_\alpha = \frac{f(x)-f(\alpha)}{x-\alpha}$, is called the rate of increase.*

The following corollary is an equivalent version of the above Theorem.

Proposition 5.2. *A function $f : I \longrightarrow \mathbb{R}$ is convex if and only if for all $\alpha \in I$ the function φ_α is increasing.*

Proof. Suppose that f is convex, let us show the growth of φ_α.

Let $(x, y, z) \in I^3$ be such that $x < y$ and $z \in [x, y]$, there exists $\lambda \in [0, 1]$ such that $z = \lambda x + (1-\lambda) y$

Indeed we can éwrite $\lambda = \frac{y-z}{y-x}$, so

$$\begin{aligned} f(z) \leqslant \lambda f(x) + (1-\lambda) f(y) \quad & \Longleftrightarrow f(z) - f(y) \leqslant \lambda \left(f(x) - f(y)\right) \\ \Longleftrightarrow f(z) - f(y) \leqslant \frac{y-z}{y-x} \left(f(x) - f(y)\right) \quad & \Longleftrightarrow \frac{f(y)-f(z)}{y-z} \geqslant \frac{f(y)-f(x)}{y-x} \\ \Longleftrightarrow \varphi_x(y) \leqslant \varphi_z(y) \text{ because } y - z < 0, & \end{aligned}$$

on the other hand $1 - \lambda = \frac{z-x}{y-x}$ then

$$\begin{aligned} f(z) \leqslant \lambda f(x) + (1-\lambda) f(y) \quad & \Longleftrightarrow f(z) - f(x) \leqslant (1-\lambda) \left(f(y) - f(x)\right) \\ \Longleftrightarrow f(z) - f(x) \leqslant \frac{z-x}{y-x} \left(f(y) - f(x)\right) \quad & \Longleftrightarrow \frac{f(z)-f(x)}{z-x} \leqslant \frac{f(y)-f(x)}{y-x} \\ \Longleftrightarrow \varphi_x(z) \leqslant \varphi_x(y) \text{ because } z - x > 0, & \end{aligned}$$

hence the growth of φ_α.

During the proof, we verified the inequality of the three slopes. □

Conversely. Suppose that the application φ_α is increasing for all $\alpha \in I$, let us show the convexity of f.

Let $x, y, z \in I^3$ with $x < z < y$ and $\lambda \in [0, 1]$.

We set $z = \lambda x + (1-\lambda) y$, such that $\lambda = \frac{y-z}{y-x}$ and $1 - \lambda = \frac{z-x}{y-x}$

By growth of φ_α we have on the one hand

$$\varphi_x(y) \leqslant \varphi_z(y) \iff \frac{f(y)-f(x)}{y-x} \leqslant \frac{f(y)-f(z)}{y-z} \iff f(y)-f(z) \geqslant \frac{y-z}{y-x}\left(f(y)-f(x)\right)$$

$$\iff f(z)-f(y) \leqslant \frac{y-z}{y-x}\left(f(x)-f(y)\right) \text{ because } y-z<0 \iff f(z)-f(y) \leqslant \lambda\left(f(x)-f(y)\right)$$

$$\iff f(z) \leqslant \lambda f(x) + (1-\lambda) f(y) \iff f(\lambda x + (1-\lambda) y) \leqslant \lambda f(x) + (1-\lambda) f(y)$$

On the other hand

$$\varphi_x(z) \leqslant \varphi_x(y) \iff \frac{f(z)-f(x)}{z-x} \leqslant \frac{f(y)-f(x)}{y-x}$$

$$\iff f(z)-f(x) \leqslant \frac{z-x}{y-x}\left(f(y)-f(x)\right) \text{ because } z-x>0$$

$$\iff f(z)-f(x) \leqslant (1-\lambda)\left(f(y)-f(x)\right)$$

$$\iff f(z) \leqslant \lambda f(x) + (1-\lambda) f(y)$$

$$\iff f(\lambda x + (1-\lambda) y) \leqslant \lambda f(x) + (1-\lambda) f(y),$$

Hence the convexity of f. □

Theorem 5.23 (Jensen's inequality)**.** *Let f be a convex function defined from I into $\mathbb{R}$. For any $(x_1, x_2, \ldots, x_n) \in I$ and for any family $(\lambda_1, \lambda_2, \ldots \lambda_n) \in \mathbb{R}$ such that $\sum\limits_{i=1}^{n} \lambda_i = 1$ we have the following convexity inequality:*

$$f\left(\sum_{i=1}^{n} \lambda_i x_i\right) \leqslant \sum_{i=1}^{n} \lambda_i f(x_i)$$

Proof. The proof is done by recurrence for all $n \in \mathbb{N}$

For $n = 2$, we have by convexity

$$f\left(\lambda_1 x_1 + \lambda_2 x_2\right) \leqslant \lambda_1 f(x_1) + \lambda_2 f(x_2)$$

Let us suppose that the inequality is true for $n \in \mathbb{N} \setminus \{1, 2\}$, let us show that this implies that it is true for $n+1$.

Let $(x_1, x_2, \ldots, x_{n+1}) \in I$ and $(\lambda_1, \lambda_2, \ldots \lambda_{n+1}) \in \mathbb{R}$ be such that $\sum\limits_{i=1}^{n+1} \lambda_i = 1$

If $\sum\limits_{i=1}^{n} \lambda_i = 0$, then $\lambda_1 = \lambda_2 = \ldots = \lambda_n = 0$ and $\lambda_{n+1} = 1$, inequality is true.

Now let $\sum_{i=1}^{n}\lambda_i = S$, we can write

$$\sum_{i=1}^{n+1}\lambda_i x_i = S\frac{\sum_{i=1}^{n}\lambda_i x_i}{S} + \lambda_{n+1}x_{n+1},\ \text{with } S \text{ and } \lambda_{n+1} \text{ are positive, with sum equal to 1.}$$

By the convexity of f

$$f\left(\sum_{i=1}^{n+1}\lambda_i x_i\right) \leqslant S \times f\left(\frac{\sum_{i=1}^{n}\lambda_i x_i}{S}\right) + \lambda_{n+1} f\left(x_{n+1}\right)$$

Now, if we put for each $1 \leqslant i \leqslant n, \lambda_i' = \frac{\lambda_i}{S}, \sum_{i=1}^{n}\lambda_i' = 1$

So, by hypothesis of recurrence we have

$$f\left(\sum_{i=1}^{n}\frac{\lambda_i}{S}x_i\right) \leqslant \sum_{i=1}^{n}\frac{\lambda_i}{S}f(x_i).$$

Eventually

$$f\left(\sum_{i=1}^{n+1}\lambda_i x_i\right) \leqslant \underbrace{S \times \sum_{i=1}^{n}\frac{\lambda_i}{S}f\left(x_i\right) + \lambda_{n+1}f\left(x_{n+1}\right)}_{\sum_{i=1}^{n+1}\lambda_i f(x_i)}$$

which ends the proof. □

5.2.3 Continuous convex functions

Definition 5.17. *A function $f : I \longrightarrow \mathbb{R}$ is continuous at a point α of I if:*

$$\forall \varepsilon > 0, \exists \eta > 0, \forall x \in I; |x - \alpha| < \eta \Longrightarrow |f(x) - f(\alpha)| < \varepsilon$$

Theorem 5.24. *If the function $f : I \longrightarrow \mathbb{R}$ is convex on I, then it admits a right derivative à and a left derivative à at any point in the interior of I.noté $\overset{\circ}{I}$. f is continuous on $\overset{\circ}{I}$.*

Proof. Let the application be $\varphi_\alpha : I \setminus \{\alpha\} \longrightarrow \mathbb{R}.\forall \alpha \in I$ for all $x \in]a, \alpha[$and $y \in]\alpha, b[$

Then the limit à left and à right in α of $\varphi_\alpha$exists:

$$\lim_{x\to\alpha^-}\varphi_\alpha(x) = \lim_{x\to\alpha^-}\frac{f(x)-f(\alpha)}{x-\alpha} = f_g'(\alpha)$$
$$\lim_{y\to\alpha^+}\varphi_\alpha(y) = \lim_{x\to\alpha^+}\frac{f(y)-f(\alpha)}{y-\alpha} = f_d'(\alpha)$$

hence the continuity of f.
Let $\alpha \in \mathring{I}$ then $\forall x \in]a,\alpha[$ and $y \in]\alpha, b[$ on has

$$\frac{f(x)-f(\alpha)}{x-\alpha} \leqslant \frac{f(y)-f(\alpha)}{y-\alpha}$$

When x tends to α^- and y tends to α^+, we have inequality

$$f_g'(\alpha) \leq f_d'(\alpha)$$

Let $\beta \in I^0$, with $\alpha < \beta, \forall x \in]\alpha,\beta[$ we have:

$$\frac{f(x)-f(\alpha)}{x-\alpha} \leq \frac{f(\beta)-f(\alpha)}{\beta-\alpha} \leq \frac{f(x)-f(\beta)}{x-\beta}$$

$$\Rightarrow \lim_{x\to\alpha^-}\frac{f(x)-f(\alpha)}{x-\alpha} \leq \frac{f(\beta)-f(\alpha)}{\beta-\alpha} \leq \lim_{x\to\alpha^-}\frac{f(x)-f(\beta)}{x-\beta}$$
$$\Leftrightarrow f_g'(\alpha) \leq \frac{f(\beta)-f(\alpha)}{\beta-\alpha} \leq f_g'(\beta)$$

SO

$$f_g'(\alpha) \leq f_g'(\beta)$$

hence the growth of f_g' on $\mathring{I}$
Let $\beta \in I^0$, with $\alpha <\beta, \forall y \in]\alpha,\beta[$ we have:

$$\frac{f(y)-f(\alpha)}{y-\alpha} \leq \frac{f(\beta)-f(\alpha)}{\beta-\alpha} \leq \frac{f(y)-f(\beta)}{y-\beta}$$

$$\Rightarrow \lim_{y\to\alpha^+}\frac{f(y)-f(\alpha)}{y-\alpha} \leq \frac{f(\beta)-f(\alpha)}{\beta-\alpha} \leq \lim_{y\to\alpha^+}\frac{f(y)-f(\beta)}{y-\beta}$$
$$\Leftrightarrow f_d'(\alpha) \leq \frac{f(\beta)-f(\alpha)}{\beta-\alpha} \leq f_d'(\beta).$$

So

$$f'_d(\alpha) \leq f'_d(\beta)$$

hence the growth of f'_d on $\overset{\circ}{I}$. $\square$

Remark 5.10. *If f is continuous on* $I \not\Rightarrow f$ *is convex on* I

If f is derivable on $I \not\Rightarrow f$ *is convex on* I

5.2.4 Convex derivable functions

Definition 5.18. *A function* $f : I \longrightarrow \mathbb{R}$ *is derivable in* α *of* $\overset{\circ}{I}$. *if and only if:*

$$\lim_{\alpha \to 0} \frac{f(x) - f(\alpha)}{x - \alpha} \textit{exists}$$

Definition 5.19. *Let* f *be a function defined on the open interval* I *in* $\mathbb{R}$. *Let* α *be a point of* I. *We say that* α *is a:*

- local maximum of f if

$$\exists \varepsilon > 0, |x - \alpha| \leqslant \varepsilon \implies f(x) \leqslant f(\alpha)$$

- local minimum of f if

$$\exists \varepsilon > 0, |x - \alpha| \leqslant \varepsilon \implies f(x) \geqslant f(\alpha)$$

Proposition 5.3. *Let* f *be a function defined on the open interval* I *in* $\mathbb{R}$. *If* f *has a local extremum (maximum or minimum) at a point* α *of* I *and if* f *is differentiable at* α, *then* $f'(\alpha) = 0$.

Proof. If α is a local minimum of f, then it is a local maximum of $-f$, without loss of generality we can assume that α is a local maximum.

$$\exists \varepsilon > 0, \forall x \in [\alpha - \varepsilon, \alpha + \varepsilon] : f(x) \leqslant f(\alpha)$$

so for all x in $[\alpha - \varepsilon, \alpha[$

$$\frac{f(x) - f(\alpha)}{x - \alpha} \geqslant 0 \text{ thus } \lim_{x \to \alpha^-} \frac{f(x) - f(\alpha)}{x - \alpha} = f'(\alpha) \geqslant 0$$

for all x in $]\alpha, \alpha+\varepsilon]$

$$\frac{f(x)-f(\alpha)}{x-\alpha} \leqslant 0 \text{ so } \lim_{x\to\alpha^+} \frac{f(x)-f(\alpha)}{x-\alpha} = f'(\alpha) \leqslant 0$$

which implies that $f'(\alpha) = 0$. $\square$

Theorem 5.25. *Let a function $f : I \longrightarrow \mathbb{R}$ be differentiable and convex on I then* $\forall \alpha \in I, \forall x \in I$

$$f(x) \geq f(\alpha) + f'(\alpha)(x-\alpha)$$

Proof. Let f be a convex and differentiable function on I, or $\alpha \in I$, for $x \in I$, let

$$g(x) = f(x) - f(\alpha) - f'(\alpha)\,(x-\alpha)$$

then g is differentiable on I and:

$$\forall x \in I, g'(x) = f'(x) - f'(\alpha).$$

or, f' is increasing on I, we therefore obtain the table of following variations:

x	$x \leq \alpha$	$x \geq \alpha$
$g'(x)$	$-$	$+$
$g(x)$	$\searrow$	$\nearrow$

which proves that $\forall x \in I, g(x) \geqslant 0$, so $\forall x \in I, f(x) \geq f(\alpha) + f'(\alpha)(x-\alpha)$ $\square$

Proposition 5.4. *Let $f : I \longrightarrow \mathbb{R}$ a continuous and derivable function on I, then it is convex if and only if f' is increasing.*

Proof. Let us start with the necessary condition (growth of f'). Let $x, y \in I$ be two points such that $x < y$. For all $z \in [x, y]$

$$\frac{f(z)-f(x)}{z-x} \leqslant \frac{f(z)-f(y)}{z-y}$$

making z tend towards x

$$f'(x) \leqslant \frac{f(x)-f(y)}{x-y}$$

similarly, making z tend towards y

$$\frac{f(x)-f(y)}{x-y} \leqslant f'(y)$$

so $f'(x) \leqslant f'(y)$.

Let us now show the sufficient condition (convexity of f).

For $\lambda \in [0,1]$ and $a = \lambda x + (1-\lambda)\, y$. Let us apply Mean value theorem theorem on the two intervals $[x,a]$ and $[a,y]$, there exist $c_1 \in \,]x,a[$ and $c_2 \in \,]a,y[$ such that :

$$\frac{f(x)-f(a)}{x-a} = f'(c_1) \text{ and } \frac{f(y)-f(a)}{y-a} = f'(c_2)$$

the function f' being increasing, we therefore have

$$\frac{f(x)-f(a)}{x-a} \leqslant \frac{f(y)-f(a)}{y-a},$$

this leads to $f(a) \leqslant \lambda f(x) + (1-\lambda) f(y)$. □

Corollary 5.4. *Let the function $f : I \longrightarrow \mathbb{R}$ be twice differentiable and continuous on an interval I, then it is convex if $f''(x) > 0$ for all x in I.*

Proof. This is an immediate consequence of the proposition proved (1.3).

$$f \text{ convex on } I \iff f' \text{ is increasing on } I \iff f'' \geqslant 0 \text{ on } I$$

□

Example 5.25. *Case of power functions $f : x \longmapsto x^h$ for $h \in \,]0,+\infty[$. f is of class C^∞ and we have:*

$$f'(x) = hx^{h-1} \text{ increasing } \forall x > 0$$

And

$$f''(x) = h\,(h-1)\, x^{h-2} \;\geqslant 0 \;\forall x > 0$$

hence

$$f \text{ convex} \iff h \leqslant 0 \text{ or } h \geqslant 1$$

$$f \text{ concave} \iff h \in [0,1]$$

CHAPTER

6

ELEMENTARY FUNCTIONS

Introduction

In this chapter, we propose to introduce the so-called Elementary functions.

$$e^x, \log x, a^x, \sin x, \cos x.$$

The reader is already familiar with these functions but this acquaintance is based on a treatment which was essentially based on intuitive and less rigorous geometrical considerations. Even the question of existence was ignored.

We shall base the study of these functions on the set of real numbers as a complete ordered field, the notion of limit and the convergence of series. Starting from the definitions of these functions, their basic properties will be studied. It is very important to notice here that there is many ways to introduce exponential and logarithm functions. We focus only on two approaches.

6.1 First approach

6.1.1 Logarithm

Theorem 6.1. *There exists a unique function,* $\ln :]0, +\infty[\to \mathbb{R}$ *such that :*

$$\ln'(x) = \frac{1}{x} \quad \textit{(for all } x > 0\Big) \quad \textit{and} \quad \ln(1) = 0.$$

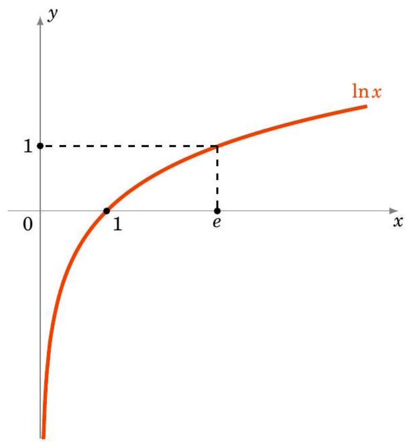
This function verifies (for all $a, b > 0$) :

1. $\ln(a \times b) = \ln a + \ln b$,
2. $\ln\left(\frac{1}{a}\right) = -\ln a$,
3. $\ln(a^n) = n \ln a$, (*for all* $n \in \mathbb{N}$)
4. *ln is a continuous function, increasing and define a (one to one) bijection from* $]0, +\infty[$ on $\mathbb{R}$,
5. $\lim_{x\to 0} \frac{\ln(1+x)}{x} = 1$,
6. *the function* $\ln$ *is concave and* $\ln x \leqslant x - 1$ *(for all $x > 0$).*

Proof. Integral theory ensures the existence and uniqueness : $\ln(x) = \int_1^x \frac{1}{t} dt$.

1. Set $f(x) = \ln(xy) - \ln(x)$ where $y > 0$ is fixed. Then $f'(x) = y \ln'(xy) - \ln'(x) = \frac{y}{xy} - \frac{1}{x} = 0$. Thus, the derivative of $x \mapsto f(x)$ is equal to zero, therefore, the function is constant et equal to $f(1) = \ln(y) - \ln(1) = \ln(y)$. So $\ln(xy) -$ $\ln(x) = \ln(y)$.
2. From a side: $\ln\left(a \times \frac{1}{a}\right) = \ln a + \ln \frac{1}{a}$, but from the second side: $\ln\left(a \times \frac{1}{a}\right) = \ln(1) = 0$. So $\ln a + \ln \frac{1}{a} = 0$.
3. By induction.
4. In is differentiable, so continuous and $\ln'(x) = \frac{1}{x} > 0$ therefore ln is increasing. Since $\ln(2) > \ln(1) = 0$ then $\ln(2^n) = n \ln(2) \to +\infty$ (when $n \to +\infty$). Thus $\lim_{x\to+\infty} \ln x = +\infty$. From $\ln x = -\ln \frac{1}{x}$ we deduce $\lim_{x\to 0} \ln x = -\infty$. Using the theorem on increasing and continuous function we get that, $\ln :]0, +\infty[\to \mathbb{R}$ is bijective (one-to-one function).
5. $\lim_{x\to 0} \frac{\ln(1+x)}{x}$ is the derivative of ln at the point $x_0 = 1$, so it exists and equals $\ln'(1) = 1$.
6. $\ln'(x) = \frac{1}{x}$ is decreasing, so the function ln is concave. Let $f(x) = x - 1 - \ln x$; $f'(x) = 1 - \frac{1}{x}$. f attains its minimum at $x_0 = 1$. Then $f(x) \geqslant f(1) = 0$. So $\ln x \leqslant x - 1$.

□



Remark 6.1. *ln is called natural logarithm function, which is characterized by* $\ln(e) = 1$.

Definition 6.1. *Given a positive real number a such that $a \neq 1$, the logarithm of a positive real number x with respect to base a is the exponent by which a must be raised to yield x. In other words, the logarithm of x to base a is the unique real number y such that $a^y = x$..*

The logarithm is denoted $\log_a$ (pronounced as "the logarithm of x to base a", "the base-a logarithm of x", or most commonly "the log, base a, of x").

An equivalent and more succinct definition is that the function $\log_a$ is the inverse function to the function $x \mapsto a^x$. More precisely we define

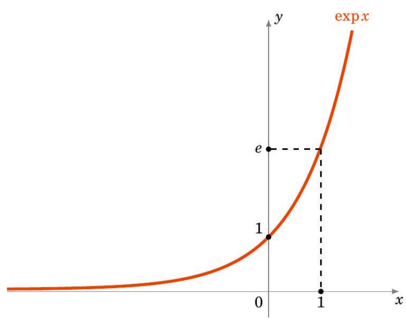
$$\log_a(x) = \frac{\ln(x)}{\ln(a)}$$

such that $\log_a(a) = 1$.

Remark 6.2. *If $a = 10$ we obtain the decimal logarithm $\log_{10}$ that verifies $\log_{10}(10) = 1$ (and so $\log_{10}(10^n) = n$). For some purposes we use : $x = 10^y \iff y = \log_{10}(x)$ in computer sciences $\log_2(2^n) = n$ is widely used.*

6.1.2 Exponential

Definition 6.2. *The inverse function of* $\ln :]0, +\infty[\to \mathbb{R}$ *is called exponential function, noted* $\exp : \mathbb{R} \to]0, +\infty[$.



For $x \in \mathbb{R}$.

Proposition 6.1. *Exponential function verifies the following properties:*

- $\exp(\ln x) = x$ *for all* $x > 0$ *and* $\ln(\exp x) = x$ *for all* $x \in \mathbb{R}$

– $\exp(a+b) = \exp(a) \times \exp(b)$

– $\exp(nx) = (\exp x)^n$

– $\exp : \mathbb{R} \to]0, +\infty[$ *is continuous, increasing where* $\lim_{x\to-\infty} \exp x = 0$ *and* $\lim_{x\to+\infty} \exp = +\infty$.

- *Exponential function is differentiable and* $\exp' x = \exp x$, *for all* $x \in \mathbb{R}$. *It is convex and* $\exp x \geqslant 1 + x$

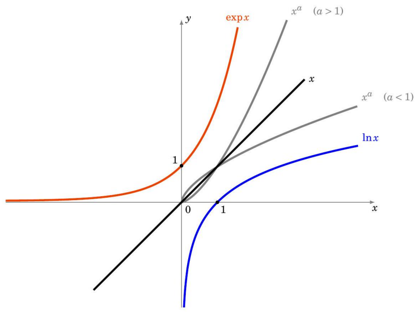
Proof. Exponential function is the natural logarithm inverse.

□

Remark 6.3. *The exponential function is the unique function which verifies* $\exp'(x) = \exp(x)$ *(for all* $x \in \mathbb{R}$ *) and* $\exp(1) = e$, *where* e *satisfies* $\ln e = 1$.

Proposition 6.2.

$$\lim_{x\to+\infty} \frac{\ln x}{x} = 0 \quad et \quad \lim_{x\to+\infty} \frac{\exp x}{x} = +\infty$$



Proof. 1. One has from previous $\ln x \leqslant x - 1$ (for all $x > 0$). Therefore $\ln x \leqslant x$ thus $\frac{\ln\sqrt{x}}{\sqrt{x}} \leqslant 1$. Then

$$0 \leqslant \frac{\ln x}{x} = \frac{\ln\left(\sqrt{x}^2\right)}{x} = 2\frac{\ln\sqrt{x}}{x} = 2\frac{\ln\sqrt{x}}{\sqrt{x}}\frac{1}{\sqrt{x}} \leqslant \frac{2}{\sqrt{x}}$$

which implies that $\lim_{x\to+\infty} \frac{\ln x}{x} = 0$.

2. We have $\exp x \geqslant 1 + x$ (for all $x \in \mathbb{R}$). So $\exp x \to +\infty$ (when $x \to +\infty$).

$$\frac{x}{\exp x} = \frac{\ln(\exp x)}{\exp x} = \frac{\ln u}{u}.$$

We conclude using (1). □

6.2 Second approach

Exponential functions

The power series

$$1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots \tag{6.1}$$

is everywhere convergent for real x. We proceed now to examine in detail the function represented by this series.

Definition 6.3. *The function represented by the power series (6.1) is called the Exponential function, denoted, provisionally, by $expx$. Thus*

$$\begin{aligned} expx &= 1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots \\ exp(0) &= 1 \end{aligned} \tag{6.2}$$

and

$$exp(1) = 1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{n!} + \ldots \tag{6.3}$$

The series on the right hand side of (6.3) converges to a number which lies between 2 *and* 3*. This number is denoted by* e*, the Exponential base and is the same number as represented by*

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$$

Thus $exp(1) = e$*.*

The Additional Formula

The function $expx$, defined by (6.2) is continuous and differentiable any number of times, for every x.

By differentiation, we get

$$\begin{aligned} exp'(x) &= expx \\ exp''(x) &= expx \\ &\vdots \\ exp^{(n)}(x) &= expx \end{aligned}$$

Further we state (justification may be seen expanding by Taylor's Theorem)* that

$$exp\,(x_1 + x_2) = exp\,(x_1) \cdot exp\,(x_2)$$

$*expx = exp\,(x_1) + \frac{exp(x_1)}{1!}\,(x - x_1) + \ldots$, for all values of x and x_1.

Replacing x by $x_1 + x_2$, we get

$exp\,(x_1 + x_2) \;=\; exp\,(x_1)\left\{1 + \frac{x_2}{1!} + \frac{x_2^2}{2!} + \ldots\right\} \;=\; exp\,(x_1) \cdot exp\,(x_2)$ This formula is called the Addition formula for the exponential function. It gives further

$$\begin{aligned} exp\,(x_1 + x_2 + x_3) &= exp\,(x_1 + x_2) \cdot exp\,(x_3) \\ &= exp\,(x_1) \cdot exp\,(x_2) \cdot exp\,(x_3) \end{aligned}$$

and repetition of the process gives, for any positive integer q,

$$exp\,(x_1 + x_2 + \ldots + x_q) = exp\,(x_1) \cdot exp\,(x_2) \ldots exp\,(x_q) \tag{6.4}$$

If $x_1 = x_2 = x_3 = \ldots = x_q = x$, we get

$$exp(qx) = exp(x)^q \tag{6.5}$$

Hence for $x = 1$,

$$exp(q) = \{exp(1)\}^q = e^q, \text{ for any positive integer } q$$

But since $exp(0) = 1$, therefore the above relation holds for $q = 0$ also.

Hence $exp(q) = e^q$ holds for all integers ≥ 0.

Again replacing each x by p/q in (6.5), we get

$$exp\left(q\frac{p}{q}\right) = \left\{exp\left(\frac{p}{q}\right)\right\}^q \text{ for positive integers, } p, q$$

or

$$exp(p/q) = \{exp(p)\}^{1/q} = e^{p/q} \quad [\because exp(p) = e^p]$$

Hence $exp(m) = e^m$, for all rational numbers $m \geq 0$.

For any positive irrational number ξ there always exists a sequence (x_n) of positive rational terms, converging to ξ.

Now for each n

$$exp\,(x_n) = e^x n.$$

When $n \to +\infty$, the left hand side tends to $exp(\xi)$, and the right hand side to e^{ξ}, so that we get

$$exp(\xi) = e^{\xi}$$

$$expx = e^{x}, \text{ for real } x \geq 0 \tag{6.6}$$

Again by Addition formula,

$$expx \cdot exp(-x) = exp(x - x) = exp(0) = 1 \tag{6.7}$$

Thus we conclude that $expx \neq 0$, for any real x, and that for $x \geq 0$,

$$exp(-x) = \frac{1}{expx} = \frac{1}{e^{x}} = e^{-x},$$

Consequently, $expx = e^{x}$ holds for all real x.

Monotonicity

By definition

$$expx > 0, \forall x > 0$$

so that from (6.7) it follows that

$$exp(-x) > 0, \quad \forall x > 0$$

Hence $expx > 0$, for all real x.
Again by definition, for real x,

$$expx \to +\infty, \text{ as } x \to +\infty$$

Hence (6.7) shows that

$$expx \to 0 \text{ as } x \to -\infty$$

Also by definition,

$$0 < x_1 < x_2 \Rightarrow exp\,(x_1) < exp\,(x_2)$$

Also it follows from (6.7) that

$$exp\,(-x_2) < exp\,(-x_1)\,,\ \text{when}\ -x_2 < -x_1 < 0$$

Hence the function exp is strictly increasing from 0 to $+\infty$ on the whole real line.

Note. By definition $e^x > \frac{x^{n+1}}{(n+1)!}$, for $x > 0$, so that $x^n e^{-x} < \frac{(n+1)!}{x}$.

$\therefore \lim_{x \to +\infty} x^n e^{-x} = 0$, for all n

This fact we express by saying that e^x tends to $+\infty$ "faster" than any power of x, as $x \to +\infty$.

Logarithmic functions (base e)

Since the exponential function exp is strictly increasing on the set $\mathbb{R}$ of real numbers (i.e., $exp : \mathbb{R} \to \mathbb{R}^+$is one-one, onto), it has inverse function ln (or $\log_e$) which is also strictly increasing and whose domain of definition is $\mathbb{R}^+ (= exp(\mathbb{R}))$, the set of positive reals. Thus ln is defined by

$$exp\{ln(y)\} = y, (y > 0)$$

or

$$ln\{expx\} = x, (x \text{ real }) \tag{6.8}$$

or equivalently, for any real x,

$$\left.\begin{array}{r} expx = y \Rightarrow ln(y) = x \\ e^x = y \Rightarrow \log_e y = x \end{array}\right\}$$

Thus the logarithmic function ln (or $\log_e$) is defined for positive values only of the variable.

By definition,

$$\left.\begin{array}{c} exp(-x) = \frac{1}{y} \Rightarrow ln\left(\frac{1}{y}\right) = -x = -ln(y) \\ exp(0) = 1 \Rightarrow ln(1) = 0 = \log_e 1 \\ exp(1) = e \Rightarrow ln(e) = 1 = \log_e e \end{array}\right\}$$

Again

$\because$

$$expx \to +\infty \text{ as } x \to +\infty$$

and

$expx \to 0$ as $x \to -\infty$

$ln(x) \to +\infty$ as $x \to +\infty$

$ln(x) \to -\infty$ as $x \to 0$

$\therefore$ Writing $u = exp\,(x_1)\,, v = exp\,(x_2)$ or $ln(u) = x_1, ln(v) = x_2$ in (6.4), we get

$exp\,(x_1 + x_2) = uv$

$\Rightarrow ln(uv) = x_1 + x_2 = ln(u) + ln(v)$

which is a familiar property of the logarithmic function and which makes logarithms a useful tool for computation.

Since the function exp is differentiable, therefore, its inverse function ln is also differentiable.

Hence differentiating (6.8), we get

$$ln'\{expx\} \cdot expx = 1$$

Writing $expx = y$, we get

$$ln'(y) = \frac{1}{y}$$

which implies that

$$ln(y) = \int_1^y \frac{dx}{x} \tag{6.9}$$

Quite often (6.9) is taken as the definition of the logarithmic function and thus the starting point of the theory of the logarithmic and the exponential functions.

Note. In theoretical investigations, it is always more convenient to use the so-called natural logarithms, that is to say, those with the base e. Hence in our further discussion, $\log x$ shall always stand for $ln(x)$ or $\log_e x$.

Generalised Power Functions

The meaning of a^x is well understood when a is any positive real number and x is any rational number. We shall now give a meaning to a^x when x is any real number whatsoever. We define thus:

Definition 6.4. $a^x = exp(x \log a)$, *for all* x *and* $a > 0$.

Evidently the range of a^x is the set $\mathbb{R}^+$of positive reals, i.e.,

$$a^x > 0, \forall x$$

Therefore $a^x \cdot a^y = exp(x \log a) \cdot exp(y \log a)$

$$= exp\{(x+y)\log a\} = a^{x+y}$$

Thus $\quad a^x \cdot a^y = a^{x+y}$

Let us now verify that this definition of a^x is consistent with that already known to us for x, an integer or a rational number.

(i) Let $x = n$, a positive integer. Therefore

$$\begin{aligned} a^n &= exp(n \log a) = exp[\log a + \log a + \ldots n \text{ times }] \\ &= exp(\log a) \cdot exp(\log a) \ldots n \text{ times} \\ &= a \cdot a \ldots n \text{ times} \end{aligned}$$

(ii) Now let $x = -n, n$ being a positive integer.

Therefore

$$\begin{aligned} a^{-n} &= exp(-n \log a) \\ &= exp[(-\log a) + (-\log a) + \ldots n \text{ times }] \\ &= exp\left[\log \frac{1}{a} + \log \frac{1}{a} + \ldots n \text{ times }\right] \\ &= exp\left(\log \frac{1}{a}\right) \cdot exp\left(\log \frac{1}{a}\right) \ldots n \text{ times} \\ &= \frac{1}{a} \cdot \frac{1}{a} \ldots n \text{ times} \end{aligned}$$

Thus, $exp(x \log a)$ has the same meaning as a^x when x is an integer.

(iii) Let now $x = p/q$, where p, q are integers, and q is positive.

Now

$$exp\left(\frac{p}{q} \log a\right) = a^{p/q}$$

$$\left[exp\left(\frac{p}{q} \log a\right)\right]^q = a^p = exp(p \log a)$$

so that $exp\left(\frac{p}{q} \log a\right)$ is q th root of $exp(p \log a)$.

Thus, $a^{p/q}$ is a q th root of a^p.

Hence, the definition holds good when x is a rational number.

Thus, the above definition of a^x agrees with what is already known to us about a^x.

Logarithmic Functions (any base)

Definition 6.5. $a^x = y \Leftrightarrow \log_a y = x.$

Since y is always positive, therefore the logarithmic function, $\log_a$, is defined for positive values only of the variable.

Evidently

$$a^{-x} = \frac{1}{y}$$
$$\log_a \frac{1}{y} = -x = -\log_a y$$

Also, from definition,

It may be easily shown that

$$\begin{aligned}
&\log_a 1 = 0, \log_a a = 1 \\
&\log_a x + \log_a y = \log_a(xy) \\
&\log_a x - \log_a y = \log_a(x/y) \\
&\log_a x^y = y \log_a x \\
&\log_b x \cdot \log_a b = \log_a x \\
&\log_b a \cdot \log_a b = 1
\end{aligned}$$

6.2.1 Trigonometric functions

We are now in a position to introduce rigorously the circular functions, employing purely the arithmetical methods. For this purpose, we consider the power series, everywhere convergent (absolutely and uniformly) and the functions represented by them.

Definition.

$$C(x) = 1 - \frac{x^2}{2!} + \frac{x^2}{2!} - \ldots + (-1)^n \frac{x^{2n}}{(2n)!} + \ldots \quad \forall x$$
$$S(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \ldots \quad \forall x$$

Each of these series represents a function everywhere continuous and differentiable any number of times in succession. The properties of these functions will be established, taking as starting point their expansions in series form, and it will be seen finally that these coincide with the functions $\cos x$ and $\sin x$ with which we are familiar from elementary studies, i.e., $C(x) = \cos x$ and $S(x) = \sin x$.

Properties of the Functions $(C(x), S(x))$

(i) The functions $C(x)$ and $S(x)$ are continuous and derivable for all x; in fact it may easily be seen that

$$C'(x) = -S(x) \text{ and } S'(x) = C(x)$$

(ii) From definitions,

$$\begin{aligned}
&S(0) = 0, C(0) = 1 \\
&S(-x) = -x - \frac{(-x)^3}{3!} + \frac{(-x)^5}{5!} - \ldots + (-1)^n \frac{(-x)^{2n+1}}{(2n+1)!} + \ldots \\
&= -\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \ldots\right] = -S(x) \forall x
\end{aligned}$$

Similarly, $C(-x) = C(x) \forall x$.

(iii) The Addition Theorems. These functions, like the exponential function, satisfy simple addition theorems, by means of which they can then be further examined.

First Method. By Taylor's expansion for any two variables, x_1 and x_2 (since the two series converge everywhere absolutely).

$$\begin{aligned}
C(x_1 + x_2) &= C(x_1) + \frac{C'(x_1)}{1!} x_2 + \frac{C''(x_1)}{2!} x_2^2 + \ldots \\
&= C(x_1) - \frac{S(x_1)}{1!} x_2 - \frac{C(x_1)}{2!} x_2^2 + \frac{S(x_1)}{3!} x_2^3 + \ldots
\end{aligned}$$

As this series is absolutely convergent, we may rearrange it in any way we please. Therefore

$$\begin{aligned}
C(x_1 + x_2) &= C(x_1) \left\{1 - \frac{x_2^2}{2!} + \frac{x_2^4}{4!} - \ldots\right\} - S(x_1) \left\{x_2 - \frac{x_2^3}{3!} + \frac{x_2^5}{5!} - \ldots\right\} \\
&= C(x_1) \cdot C(x_2) - S(x_1) \cdot S(x_2)
\end{aligned}$$

Similarly,

$$S(x_1 + x_2) = S(x_1) \cdot C(x_2) + C(x_1) \cdot S(x_2)$$

Second Method. For any fixed value of x_2, consider the functions

$$\begin{aligned}
f(x_1) &= S(x_1 + x_2) - S(x_1) \cdot C(x_2) - C(x_1) \cdot S(x_2) \\
g(x_1) &= C(x_1 + x_2) - C(x_1) \cdot C(x_2) + S(x_1) \cdot S(x_2)
\end{aligned}$$

Differentiating with respect to x_1, we get

$$
\begin{aligned}
f'(x_1) &= C(x_1+x_2) - C(x_1)\cdot C(x_2) + S(x_1)\cdot S(x_2) = g(x_1)\\
g'(x_1) &= -S(x_1+x_2) + S(x_1)\cdot C(x_2) + C(x_1)\cdot S(x_2) = -f(x_1)
\end{aligned}
$$

$$
\text{therefore} \quad \frac{d}{dx_1}\left[f^2(x_1)+g^2(x_1)\right] = 2f(x_1)f'(x_1) + 2g(x_1)g'(x_1)
$$
$$
= 2f(x_1)g(x_1) - 2g(x_1)f(x_1) = 0, \forall x_1
$$

$\Rightarrow f^2(x_1)+g^2(x_1)$ is a constant, $\forall x_1$

Hence for all x_1

$$
\begin{aligned}
f^2(x_1)+g^2(x_1) &= f^2(0)+g^2(0) = 0\\
\Rightarrow \quad f(x_1) = 0, \quad g(x_1) &= 0\\
\text{therefore} \qquad C(x_1+x_2) &= C(x_1)\cdot C(x_2) - S(x_1)\cdot S(x_2)
\end{aligned}
$$

$$
\Rightarrow \quad f(x_1) = 0, g(x_1) = 0
$$

and

$$
S(x_1+x_2) = S(x_1)\cdot C(x_2) + C(x_1)\cdot S(x_2)
$$

The form of these theorems coincides with that of the addition theorems for the functions cosine and sine, with which we are clearly acquainted from an elementary standpoint. With the help of these theorems, we shall now show that the functions C and S satisfy all the other so called purely trigonometrical formulae-in fact C and S are same as the functions cosine and sine. We note, in particular:

(a) Changing x_2 to $-x_2$,

$$
\begin{aligned}
C(x_1-x_2) &= C(x_1)\cdot C(x_2) + S(x_1)\cdot S(x_2)\\
S(x_1-x_2) &= S(x_1)\cdot C(x_2) - C(x_1)\cdot S(x_2)
\end{aligned}
$$

(b) Writing $x_2 = -x_1$, we deduce that

$$
\begin{aligned}
&C^2(x_1)+S^2(x_1) = 1 \text{ or } C^2(x)+S^2(x) = 1, \forall x\\
&\Rightarrow |S(x)| \le 1, |C(x)| \le 1, \forall x
\end{aligned}
$$

(c) Replacing x_1 and x_2 by x,

$$
\begin{aligned}
C(2x) &= C^2(x) - S^2(x)\\
S(2x) &= 2S(x)\cdot C(x)
\end{aligned}
$$

Theorem 6.2. *There exists a positive number π, such that*

$$C(\pi/2) = 0 \text{ and } C(x) > 0, \text{ for } 0 \le x < \pi/2$$

Proof. Consider the interval $[0, 2]$.

We know $C(0) = 1 > 0$; we shall now show that $C(2) < 0$.

Now

$$\begin{aligned} C(2) &= 1 - \frac{2^2}{2!} + \frac{2^4}{4!} - \frac{2^6}{6!} + \dots \\ &= 1 - \frac{2^2}{2!}\left(1 - \frac{2^2}{3.4}\right) - \frac{2^6}{6!}\left(1 - \frac{2^2}{7.8}\right) - \dots \end{aligned}$$

Since the brackets are all positive, we have

$$C(2) < 1 - \frac{2^2}{2!}\left(1 - \frac{2^2}{3.4}\right) = -\frac{1}{3}$$

so that $C(2)$ is negative.

Thus, the continuous function $C(x)$ is positive at 0 and negative at 2 .

$C(x)$ vanishes at least once between 0 and 2 (by the Intermediate-value theorem). Further, since $S(x)$ is positive in $[0, 2]$, where

$$S(x) = x\left(1 - \frac{x^2}{2.3}\right) + \frac{x^5}{5!}\left(1 - \frac{x^2}{6.7}\right) + \dots$$

therefore, the derivative $(-S(x))$ of $C(x)$ is always negative for all values of x between 0 and 2 . Consequently $C(x)$ is a (strictly) monotonic decreasing function in $[0, 2]$, and can therefore, vanish at only one point in $[0, 2]$. Thus, there exists one and only one root of the equation $C(x) = 0$ lying between 0 and 2 . Denoting this root by $\pi/2$, we see that $\pi/2$ is the least positive root of the equation $C(x) = 0$. □

Clearly $C(x) > 0$, when $0 \le x < \pi/2$.

Using the above results, we deduce that

(a) $S(x) > 0$, when $0 < x \le \pi/2$.

Since the derivative of $S(x)$ is non-negative in $[0,\ \pi/2]$, therefore, $S(x)$ is a strictly monotonic increasing function. Also since $S(0) = 0$, therefore, $S(x)$ is positive for $0 < x \le \pi/2$.

(b) As $C^2(\pi/2) + S^2(\pi/2) = 1$ and $C(\pi/2) = 0$,

$\Rightarrow \quad S^2(\pi/2) = 1 \Rightarrow S(\pi/2) = \pm 1$

But, by Lagrange's Mean Value Theorem,

$S(\pi/2) - S(0) = (\pi/2)C(\alpha) > 0$, where $0 < \alpha < \pi/2$

$\Rightarrow \quad S(\pi/2) = 1$

(c) $C(\pi) = 2C^2(\pi/2) - 1 = -1$

$$S(\pi) = 2S(\pi/2)C(\pi/2) = 0$$

(d) $C(2\pi) = 1, S(2\pi) = 0.$

(e) $C(\pi/2) = 2C^2(\pi/4) - 1.$

rejecting the negative sign, as $C(\pi/4)$ is positive.

Similarly, $S(\pi/4) = 1/\sqrt{2}$

(f) It finally follows from the addition theorems that for all x,

$$\begin{aligned}
S\left(\frac{1}{2}\pi - x\right) &= C(x), \quad & C\left(\frac{1}{2}\pi - x\right) &= S(x) \\
S\left(\frac{1}{2}\pi + x\right) &= C(x), \quad & C\left(\frac{1}{2}\pi + x\right) &= -S(x) \\
S(\pi + x) &= -S(x), \quad & C(\pi + x) &= -C(x) \\
S(\pi - x) &= S(x), \quad & C(\pi - x) &= -C(x) \\
S(2\pi + x) &= S(x), \quad & C(2\pi + x) &= C(x)
\end{aligned}$$

Thus, we see that the functions $C(x)$ and $S(x)$ exactly coincide with the functions $\cos x$ and $\sin x$ respectively, and so we shall henceforth use $\cos x$ and $\sin x$ in place of $C(x)$ and $S(x)$ respectively.

The Functions $\tan x, \cot x$

The function $\tan x$ and $\cot x$ are defined as usual by the ratios

$$\tan x = \frac{\sin x}{\cos x}, \cot x = \frac{\cos x}{\sin x}$$

and as functions they, therefore, represent nothing new. The expansions in power series for these functions are also not so simple. A few of the coefficients of the expansions could be easily obtained by division, but that gives us no insight into any relationships.

Clearly $\tan x$ is defined, continuous and derivable for all values of x except those for which the denominator, $\cos x$, vanishes, which is the case for $x = \frac{1}{2}(2n+1)\pi, n$ being any integer, positive, negative or zero.

We have

$$\tan(\pi + x) = \tan x,$$

so that, $\tan x$ is a periodic function with period π.

Also we may easily show that when $x \neq \frac{1}{2}(2n+1)\pi$,

$$\frac{d}{dx}\tan x = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{1}{\cos^2 x}$$

Theorem 6.3. *Show that*

$$\lim_{x\to\frac{1}{2}\pi-0}\tan x = \infty, \lim_{x\to\frac{1}{2}\pi+0}\tan x = -\infty.$$

Proof. Let k be any positive number.

As $\lim_{x\to\pi/2}\sin x = 1, \exists\delta_1 > 0$, such that (taking $\varepsilon = \frac{1}{2}$),

$$\frac{1}{2} < \sin x, \quad \forall x \in \left[\frac{1}{2}\pi - \delta_1, \frac{1}{2}\pi + \delta_1\right]$$

Again, since, $\lim_{x\to\pi/2}\cos x = 0$, therefore, $\exists\delta_2 > 0$, such that

$$-\frac{1}{2k} < \cos x < \frac{1}{2k}, \quad \forall x \in \left[\frac{1}{2}\pi - \delta_2, \frac{1}{2}\pi + \delta_2\right]$$

As $\cos x$ is positive for $x \in [0, \pi/2[$, and negative for $x \in]\pi/2, \pi]$, we have

$$0 < \cos x < \frac{1}{2k}, \quad \forall x \in \left[\frac{1}{2}\pi - \delta_2, \frac{1}{2}\pi\right[$$
$$-\frac{1}{2k} < \cos x < 0, \quad \forall x \in \left]\frac{1}{2}\pi, \frac{1}{2}\pi + \delta_2\right]$$

Let $\delta = \min(\delta_1, \delta_2)$ therefore from (i) and (ii),

and from (i) and (iii),

$$\tan x = \frac{\sin x}{\cos x} > k, \quad \forall x \in \left[\frac{1}{2}\pi - \delta, \frac{1}{2}\pi\right[$$

$$\tan x = \frac{\sin x}{\cos x} < -k, \quad \forall x \in \left]\frac{1}{2}\pi, \frac{1}{2}\pi + \delta\right]$$

□

Inverse Trigonometric Functions $\cos^{-1} y, \sin^{-1} y, \tan^{-1} y$

We will denote the inverse trigonometric functions by

$\sin^{-1}, \cos^{-1}, \tan^{-1}, \mathrm{cotan}^{-1}$,

or:

$\sin^{inv}, \cos^{inv}, \tan^{inv}, \mathrm{cotan}^{inv}$,

or even:

arcsin , arccos , arctan , arccoth.

$\cos^{-1} y$ function

Since, as may be easily seen, $\cos x$ strictly decreases from $+1$ to -1 as x increases from

0 to π, the function cos is invertible and its inverse, denoted as $\cos^{-1}$, is a function with domain $[-1, 1]$ and range $[0, \pi]$. We write

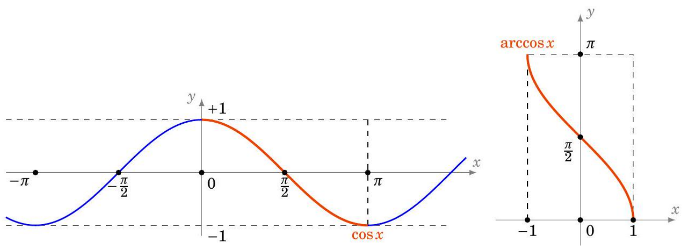
$$y = \cos x \Leftrightarrow x = \cos^{-1} y.$$

Definition 6.6. *Given y (where $-1 \leq y \leq 1$), $\cos^{-1} y$ is that x which lies between 0 and $\pi (0 \leq x \leq \pi)$ and $\cos x = y$.*

$\cos^{-1} y$ is derivable in the open interval $]-1, 1[$, with $-1/\sqrt{1-y^2}$ as its derivative. In fact, we have $\therefore \quad \frac{dx}{dy} \cdot \frac{dy}{dx} = 1$, and $x = \cos^{-1} y, y = \cos x$

$$\frac{d}{dy}\left(\cos^{-1} y\right) = \frac{1}{\frac{d}{dx}\cos x} = -\frac{1}{\sin x} = \frac{-1}{\sqrt{1-y^2}}, y \neq \pm 1.$$

$$\arccos : [-1, 1] \to [0, \pi]$$



$\sin^{-1} y$ function
Since $\sin x$ is a strictly increasing function in $[-\pi/2, \pi/2]$, with range $[-1, 1]$, therefore, the function sin is invertible and its inverse function is denoted by $\sin^{-1}$, with domain $[-1, 1]$ and range $[-\pi/2, \pi/2]$.

Also

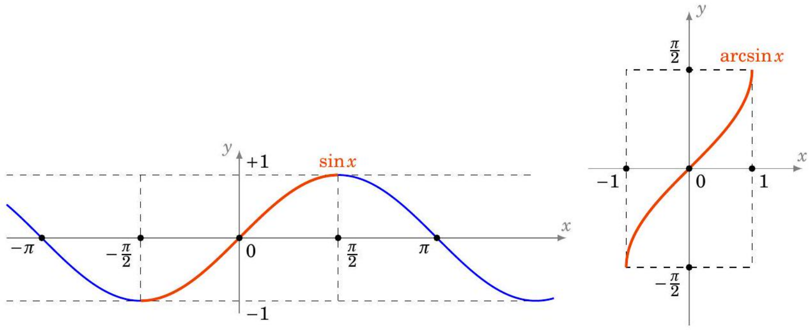
$$y = \sin x \Leftrightarrow x = \sin^{-1} y$$

Definition 6.7. *Given y where $-1 \leq y \leq 1, \sin^{-1} y$ is that x which lies between $-\pi/2$ and $\pi/2, (-\pi/2 \leq x \leq \pi/2)$, and $\sin x = y$.*

It may be shown as before that $\sin^{-1} y$ is derivable in the open interval $]\ 1, 1[$ and

$$\frac{d}{dy}\sin^{-1} y = \frac{1}{\sqrt{1-y^2}}, y \neq \pm 1$$

$$\arcsin : [-1, 1] \to \left[-\frac{\pi}{2}, +\frac{\pi}{2}\right]$$



$\tan^{-1} y$ function

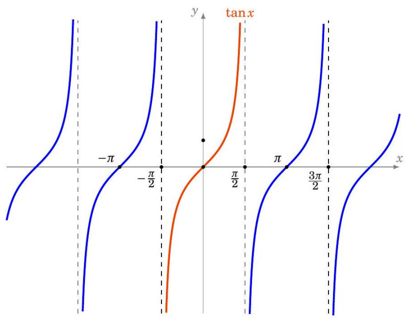
Since $\tan x$ is strictly monotonic with domain $]-\pi/2, \pi/2[$ and range $]-\infty, \infty[$, the function is invertible, we have

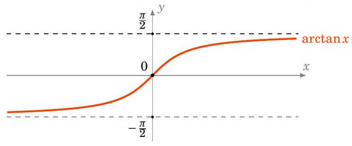
$$y = \tan x \Leftrightarrow x = \tan^{-1} y$$

so that $\tan^{-1} y$ is a function with domain $]-\infty, \infty[$ and range $]-\pi/2, \pi/2[$.

Definition 6.8. *For any number y, $\tan^{-1} y$ is that x which lies between $-\pi/2$ and $\pi/2 (-\pi/2 < x < \pi/2)$ and $\tan x = y$.*

$$\arctan : \mathbb{R} \rightarrow]-\frac{\pi}{2}, +\frac{\pi}{2}[$$





It may be seen that

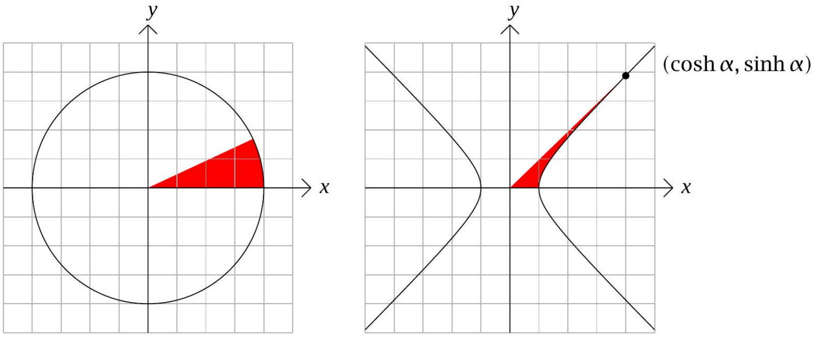
$$\frac{d}{dy} \tan^{-1} y = \frac{1}{1+y^2}, \forall y.$$

6.2.2 Hyperbolic Functions

The trigonometric functions $\cos \alpha$ and $\cos \alpha$ are defined using the unit circle $x^2 + y^2 = 1$ by measuring the distance α in the counter-clockwise direction along the circumference of the circle. The area of the sector so determined is $\frac{\alpha}{2}$, so we can equivalently say that $\cos\alpha$ and $\cos\alpha$ are derived from the unit circle $x^2+y^2=1$ by measuring off a sector (shaded red) of area $\frac{\alpha}{2}$. The other four trigonometric functions can then be defined in terms of cos and sin.

Similarly, we may define hyperbolic functions $\cosh\alpha$ and $\sinh\alpha$ from the "unit hyperbola"

$x^2-y^2=1$ by measuring off a sector (shaded red) of area $\frac{\alpha}{2}$ to obtain a point P whose x - and y-coordinates are defined to be $\cosh\alpha$ and $\sinh\alpha$.



Since at this point we do not yet know how to compute the areas of most curved regions, we must take it on faith that the six hyperbolic functions may be expressed simply in terms of the exponential function:

$$\begin{gathered}
\sinh\alpha=\frac{e^{\alpha}-e^{-\alpha}}{2}\\
\cosh\alpha=\frac{e^{\alpha}+e^{-\alpha}}{2}\\
\tanh\alpha=\frac{\sinh\alpha}{\cosh\alpha}=\frac{e^{\alpha}-e^{-\alpha}}{e^{\alpha}+e^{-\alpha}}\\
\operatorname{cotanh}\alpha=\frac{\cosh\alpha}{\sinh\alpha}=\frac{e^{\alpha}+e^{-\alpha}}{e^{\alpha}-e^{-\alpha}}\\
\operatorname{sech}\alpha=\frac{1}{\cosh\alpha}=\frac{2}{e^{\alpha}+e^{-\alpha}}\\
\operatorname{cosech}\alpha=\frac{1}{\sinh\alpha}=\frac{2}{e^{\alpha}-e^{-\alpha}}
\end{gathered}$$

Note that the domains of sinh, cosh, tanh, and sech are $(-\infty,\infty)$ and the domains of cotanh and cosech are $(-\infty,0)\cup(0,\infty)$. We can check that the point $\left(\frac{e^{\alpha}+e^{-\alpha}}{2},\frac{e^{\alpha}-e^{-\alpha}}{2}\right)$ lies on the unit hyperbola:

$$\left(\frac{e^{\alpha}+e^{-\alpha}}{2}\right)^2-\left(\frac{e^{\alpha}-e^{-\alpha}}{2}\right)^2=\frac{e^{2\alpha}+2+e^{-2\alpha}}{4}-\frac{e^{2\alpha}-2+e^{-2\alpha}}{4}=\frac{4}{4}=1$$

"Pythagorean" Identities and some laws

This gives us the first important hyperbolic function identity:

$$\cosh^2\alpha-\sinh^2\alpha\equiv 1$$

This may be used to derive two other identities relating the two other pairs of hyperbolic functions:

$$1 - \tanh^2 \alpha = \operatorname{sech}^2 \alpha \quad \text{and} \quad \operatorname{cotanh}^2 \alpha - 1 = \operatorname{cosech}^2 \alpha$$

It is clear that sinh, tanh, cotanh x and cosech are odd functions, while cosh, cotanh , and sech are even, so we have the corresponding identities:

$$\sinh(-x) = -\sinh x, \tanh(-x) = -\tanh x$$
$$\operatorname{cotanh}(-x) = -\operatorname{cotanh} x, \operatorname{cosech}(-x) = -\operatorname{cosech} x$$
$$\cosh(-x) = \cosh x, \operatorname{sech}(-x) = \operatorname{sech} x.$$

We can use the above formulas for the hyperbolic functions in terms of e^x to derive analogs of the identities for the trigonometric functions:

$$\sinh\alpha\cosh\beta = \frac{e^{\alpha}-e^{-\alpha}}{2}\frac{e^{\beta}+e^{-\beta}}{2} = \frac{\left(e^{\alpha}-e^{-\alpha}\right)\left(e^{\beta}+e^{-\beta}\right)}{4} = \frac{e^{\alpha+\beta}+e^{\alpha-\beta}-e^{-\alpha+\beta}-e^{-\alpha-\beta}}{4}$$
$$\sinh\beta\cosh\alpha = \frac{e^{\beta}-e^{-\beta}}{2}\frac{e^{\alpha}+e^{-\alpha}}{2} = \frac{\left(e^{\beta}-e^{-\beta}\right)\left(e^{\alpha}+e^{-\alpha}\right)}{4} = \frac{e^{\beta+\alpha}+e^{\beta-\alpha}-e^{-\beta+\alpha}-e^{-\beta-\alpha}}{4}$$

Adding these two products gives:

$\sinh\alpha\cosh\beta + \sinh\beta\cosh\alpha = \frac{e^{\alpha+\beta}+e^{\alpha-\beta}-e^{-\alpha+\beta}-e^{-\alpha-\beta}}{4} + \frac{e^{\beta+\alpha}+e^{\beta-\alpha}+e^{-\beta+\alpha}-e^{-\beta-\alpha}}{4} = \frac{2e^{\alpha+\beta}-2e^{-\alpha-\beta}}{4} = \frac{e^{\alpha+\beta}-e^{-\alpha-\beta}}{2} = \frac{e^{(\alpha+\beta)}-e^{-(\alpha+\beta)}}{2} = \sinh(\alpha+\beta)$

and subtracting these two products gives:

$\sinh\alpha\cosh\beta - \sinh\beta\cosh\alpha = \frac{e^{\alpha+\beta}+e^{\alpha-\beta}-e^{-\alpha+\beta}-e^{-\alpha-\beta}}{4} - \frac{e^{\beta+\alpha}+e^{\beta-\alpha}+e^{-\beta+\alpha}-e^{-\beta-\alpha}}{4} = \frac{2e^{\alpha-\beta}-2e^{-(\alpha-\beta)}}{4} = \frac{e^{\alpha-\beta}-e^{-(\alpha-\beta)}}{2} = \sinh(\alpha-\beta)$

Similarly,

$\cosh\alpha\cosh\beta = \frac{e^{\alpha}+e^{-\alpha}}{2}\frac{e^{\beta}+e^{-\beta}}{2} = \frac{\left(e^{\alpha}+e^{-\alpha}\right)\left(e^{\beta}+e^{-\beta}\right)}{4} = \frac{e^{\alpha+\beta}+e^{\alpha-\beta}+e^{\beta-\alpha}+e^{-\alpha-\beta}}{4}$

$\sinh\alpha\sinh\beta = \frac{e^{\alpha}-e^{-\alpha}}{2}\frac{e^{\beta}-e^{-\beta}}{2} = \frac{\left(e^{\alpha}-e^{-\alpha}\right)\left(e^{\beta}-e^{-\beta}\right)}{4} = \frac{e^{\alpha+\beta}-e^{\alpha-\beta}-e^{\beta-\alpha}+e^{-\alpha-\beta}}{4}$

Adding these two products gives

$\cosh\alpha\cosh\beta + \sinh\alpha\sinh\beta = \frac{e^{\alpha+\beta}+e^{\alpha-\beta}+e^{\beta-\alpha}+e^{-\alpha-\beta}}{4} + \frac{e^{\alpha+\beta}-e^{\alpha-\beta}-e^{\beta-\alpha}+e^{-\alpha-\beta}}{4}$
$= \frac{2e^{\alpha+\beta}+2e^{-\alpha-\beta}}{4} = \frac{e^{\alpha+\beta}+e^{-(\alpha+\beta)}}{2} = \cosh(\alpha+\beta)$

and subtracting them gives:

$\cosh\alpha\cosh\beta - \sinh\alpha\sinh\beta = \frac{e^{\alpha+\beta}+e^{\alpha-\beta}+e^{\beta-\alpha}+e^{-\alpha-\beta}}{4} - \frac{e^{\alpha+\beta}-e^{\alpha-\beta}-e^{\beta-\alpha}+e^{-\alpha-\beta}}{4} = \frac{2e^{\alpha-\beta}+2e^{-\alpha+\beta}}{4} = \frac{e^{\alpha-\beta}+e^{-(\alpha-\beta)}}{2} = \cosh(\alpha-\beta)$

Summarizing, we have four identities:

$\sinh(\alpha+\beta) = \sinh\alpha\cosh\beta + \sinh\beta\cosh\alpha$ $\sinh(\alpha-\beta) = \sinh\alpha\cosh\beta - \sinh\beta\cosh\alpha$

$\cosh(\alpha+\beta) = \cosh\alpha\cosh\beta + \sinh\alpha\sinh\beta$

$\cosh(\alpha-\beta) = \cosh\alpha\cosh\beta - \sinh\alpha\sinh\beta$

which are almost exactly parallel to those for the trigonometric functions and may be used to derive sum and difference formulas for the other four hyperbolic functions.

Letting $\beta = \alpha$, we get:

$\sinh 2\alpha = 2\sinh\alpha\cosh\alpha$

$\cosh 2\alpha = \cosh^2\alpha + \sinh^2\alpha = 1 + 2\sinh^2\alpha = 2\cosh^2\alpha - 1$, so

$\cosh^2\alpha = \frac{\cosh 2\alpha+1}{2}$ and $\sinh^2\alpha = \frac{\cosh 2\alpha-1}{2}$, and thus:

$\cosh\alpha = \sqrt{\frac{\cosh 2\alpha+1}{2}}$ and $\sinh\alpha = \sqrt{\frac{\cosh 2\alpha-1}{2}}$ $\cosh\frac{\alpha}{2} = \sqrt{\frac{\cosh\alpha+1}{2}}$ and $\sinh\frac{\alpha}{2} = \sqrt{\frac{\cosh\alpha-1}{2}}$

Derivatives

$$\frac{d}{dx}(\sinh x) = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) = \frac{e^x - (-e^{-x})}{2} = \frac{e^x + e^{-x}}{2} = \cosh x$$
$$\frac{d}{dx}(\cosh x) = \frac{d}{dx}\left(\frac{e^x + e^{-x}}{2}\right) = \frac{e^x + (-e^{-x})}{2} = \frac{e^x - e^{-x}}{2} = \sinh x$$

$$\frac{d}{dx}(\tanh x) = \frac{d}{dx}\left(\frac{\sinh x}{\cosh x}\right) =$$
$$\frac{\cosh x(\sinh x)' - \sinh x(\cosh x)'}{\cosh^2 x} = \frac{\cosh x\cosh x - \sinh x\sinh x}{\cosh^2 x} =$$
$$\frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x} = \operatorname{sech}^2 x$$

$$\frac{d}{dx}(\operatorname{cotanh} x) = \frac{d}{dx}\left(\frac{\cosh x}{\sinh x}\right) =$$
$$\frac{\sinh x(\cosh x)' - \cosh x(\sinh x)'}{\sinh^2 x} = \frac{\sinh x\sinh x - \cosh x\cosh x}{\sinh^2 x} =$$
$$\frac{\sinh^2 x - \cosh^2 x}{\sinh^2 x} = \frac{-1}{\sinh^2 x} = -\operatorname{cosech}^2 x$$

$$\frac{d}{dx}(\operatorname{sech} x) = \frac{d}{dx}(\cosh x)^{-1} =$$
$$(-1)(\cosh x)^{-2}(\cosh x)' = (-1)(\cosh x)^{-2}\sinh x = -\operatorname{sech} x\tanh x$$

$$\frac{d}{dx}(\operatorname{cosech} x) = \frac{d}{dx}(\sinh x)^{-1} =$$

$(-1)(\sinh x)^{-2}(\sinh x)' = (-1)(\sinh x)^{-2}\cosh x = -\operatorname{cosech} x\operatorname{cotanh} x$

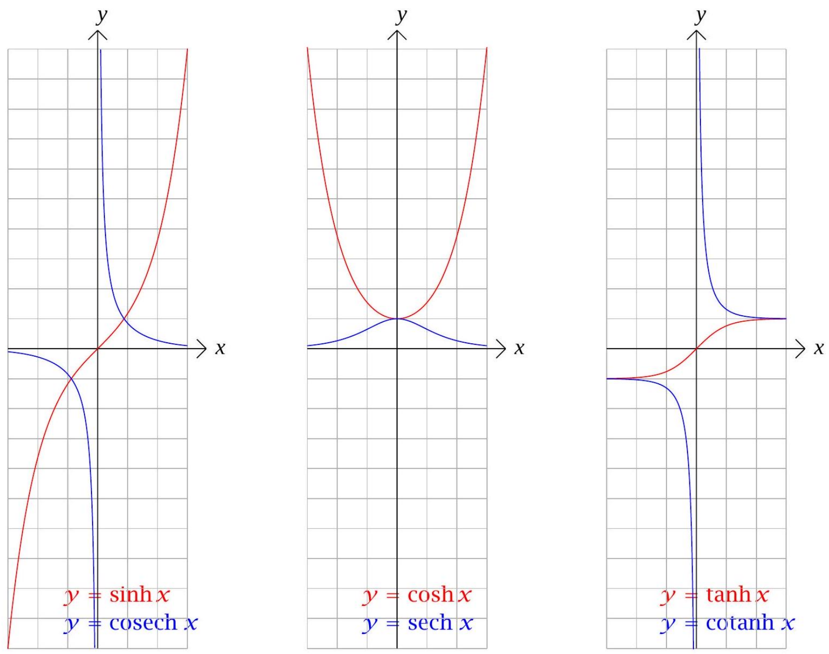
Then we can summarize them as:

$$\frac{d}{dx}(\sinh x) = \cosh x \quad \frac{d}{dx}(\cosh x) = \sinh x$$
$$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x \quad \frac{d}{dx}(\operatorname{cotanh} x) = -\operatorname{cosech}^2 x$$
$$\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x\tanh x \quad \frac{d}{dx}(\operatorname{cosech} x) = -\operatorname{cosech} x\operatorname{cotanh} x$$

The domains and ranges are summarized in the next table:

function	domain	Range
sinh	$(-\infty, \infty)$	$(-\infty, \infty)$
cosh	$(-\infty, \infty)$	$[1, \infty)$
tanh	$(-\infty, \infty)$	$(-1, 1)$
cotanh	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, -1) \cup (1, \infty)$
sech	$(-\infty, \infty)$	$(0, 1])$
cosech	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, 0) \cup (0, \infty)$

Graphs of the Hyperbolic Functions



Inverse Hyperbolic Functions

sinh, tanh, cotanh and cosech are one-to-one, but cosh and sech are not. For the purpose of defining the inverse of cosh and sech we will restrict their domains to $[0, \infty)$

We will denote the inverse hyperbolic functions by

$\sinh^{-1}, \cosh^{-1}, \tanh^{-1}, \mathrm{cotanh}^{-1}, \mathrm{sech}^{-1}$, and cosech^{-1}

or:

$\sinh^{inv}, \cosh^{inv}, \tanh^{inv}, \mathrm{cotanh}^{inv}, \mathrm{sech}^{inv}$, and cosech^{inv}

or even:

arcsinh , arccosch , arctanh , arccothh , arcsech , and arccosech .

The usual Cancellation Laws hold in the appropriate domains:

$$
\begin{aligned}
\sinh\left(\sinh^{-1}x\right) &= x & \sinh^{-1}(\sinh x) &= x\\
\cosh\left(\cosh^{-1}x\right) &= x & \cosh^{-1}(\cosh x) &= x\\
\tanh\left(\tanh^{-1}x\right) &= x & \tanh^{-1}(\tanh x) &= x\\
\operatorname{cotanh}\left(\operatorname{cotanh}^{-1}x\right) &= x & \operatorname{cotanh}^{-1}(\operatorname{cotanh} x) &= x\\
\operatorname{sech}\left(\operatorname{sech}^{-1}x\right) &= x & \operatorname{sech}^{-1}(\operatorname{sech} x) &= x\\
\operatorname{cosech}\left(\operatorname{cosech}^{-1}x\right) &= x & \operatorname{cosech}^{-1}(\operatorname{cosech} x) &= x
\end{aligned}
$$

The derivatives of the inverse hyperbolic functions may be found the same way the derivatives of the inverse trigonometric functions were found: by differentiating the left-hand Cancellation Laws above: for example let us differentiating $\sinh\left(\sinh^{-1}x\right) = x$ we get

$$
\begin{aligned}
&\cosh\left(\sinh^{-1}x\right)\left(\sinh^{-1}x\right)' = 1, \text{ so}\\
&\left(\sinh^{-1}x\right)' = \frac{1}{\cosh\left(\sinh^{-1}x\right)}.
\end{aligned}
$$

Using the identity $\cosh^2 x - \sinh^2 x = 1$ we get

$$
\begin{aligned}
&\cosh^2 x = 1 + \sinh^2 x, \text{ so}\\
&\cosh x = \sqrt{1+\sinh^2 x} \text{ and therefore}\\
&\cosh\left(\sinh^{-1}x\right) = \sqrt{1+\sinh^2\left(\sinh^{-1}x\right)} = \sqrt{1+\left(\sinh\left(\sinh^{-1}x\right)\right)^2}\\
&= \sqrt{1+x^2}
\end{aligned}
$$

Thus we have $\left(\sinh^{-1}x\right)' = \frac{1}{\sqrt{1+x^2}}$

One may similarly derive the derivatives of the other hyperbolic functions:

$$
\begin{aligned}
&\left(\cosh^{-1}x\right)' = \frac{1}{\sqrt{x-1^2}}\\
&\left(\tanh^{-1}x\right)' = \left(\operatorname{cotanh}^{-1}x\right)' = \frac{1}{1-x^2}\\
&\left(\operatorname{sech}^{-1}x\right)' = -\frac{1}{x\sqrt{1-x^2}}\\
&\left(\operatorname{cosech}^{-1}x\right)' = \frac{-1}{|x|\sqrt{1+x^2}}
\end{aligned}
$$

Example 6.1. *Solve the equation* $\sinh y = x$ *for* y *in terms of* x*. We have* $\sinh y = \frac{e^y - e^{-y}}{2} = x$*, so*

$e^y - e^{-y} = 2x$ *or* $e^y - 2x - e^{-y} = 0$*. Multiplying both sides of this equation by* e^y *we get:*

$(e^y)^2 - 2xe^y - 1 = 0$, *a quadratic equation in* e^y *which has solution*

$$e^y = \frac{-(-2x) \pm \sqrt{(-2x)^2 - 4(1)(-1)}}{2} = \frac{2x \pm \sqrt{4x^2+4}}{2} = x \pm \sqrt{x^2+1}$$

Since $x - \sqrt{x^2+1} < 0$ *and we must have* $e^y > 0$, *we get*
$e^y = x + \sqrt{x^2+1}$.
Taking logarithms of both sides of this equation, we get
$y = \ln\left(x + \sqrt{x^2+1}\right)$, *so we have*

$$\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right)$$

Similarly,

$$\cosh^{-1} x = \ln\left(x + \sqrt{x^2-1}\right) \quad \textit{and} \quad \tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

We then have
$\operatorname{cosech}^{-1} x = \sinh^{-1}\frac{1}{x} = \ln\left(\frac{1}{x} + \sqrt{\left(\frac{1}{x}\right)^2 + 1}\right) = \ln\left(\frac{1}{x} + \sqrt{\frac{1+x^2}{x^2}}\right) =$
$\ln\left(\frac{1}{x} + \frac{\sqrt{1+x^2}}{|x|}\right)$
Similarly
$\operatorname{cotanh}^{-1} x = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right)$ *and* $\operatorname{sech}^{-1} x = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right)$

CHAPTER

7

TAYLOR POLYNOMIALS, LITTLE O AND BIG O, LIMITED DEVELOPMENT IN A NEIGHBORHOOD OF A POINT

7.1 Taylor Polynomials (Brook Taylor-1685-1731)

Definition 7.1. *Let $x_0 \in [a,b]$ and suppose that $f:[a,b] \to \mathbb{R}$ is such that the derivatives $f'(x_0), f^{(2)}(x_0), f^{(3)}(x_0), \ldots, f^{(n)}(x_0)$ exist for some positive integer n. Then the polynomial*

$$\begin{aligned} P_n(x) =& f(x_0) + f'(x_0)(x-x_0) + \frac{1}{2!} f^{(2)}(x_0)(x-x_0)^2 + \\ & \cdots + \frac{1}{n!} f^{(n)}(x_0)(x-x_0)^n \end{aligned}$$

is called the nth order Taylor polynomial of f based at x_0. Using summation convention, $P_n(x)$ can be written as

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} (x-x_0)^k .$$

By construction, the derivatives of f and P_n up to order n are identical at x_0 :

$$
\begin{aligned}
P_n\left(x_0\right) &= f\left(x_0\right) \\
P_n^{(1)}\left(x_0\right) &= f^{(1)}\left(x_0\right) \\
\vdots &= \quad \vdots \\
P^{(n)}\left(x_0\right) &= f^{(n)}\left(x_0\right).
\end{aligned}
$$

It is reasonable then to suspect that $P_n(x)$ is a good approximation to $f(x)$ for points x near x_0. If $x \in [a, b]$ then the difference between $f(x)$ and $P_n(x)$ is

$$
R_n(x) = f(x) - P_n(x)
$$

and we call $R_n(x)$ the nth order remainder based at x_0. Hence, for each $x^* \in [a, b]$, the remainder $R_n\left(x^*\right)$ is the error in approximating $f\left(x^*\right)$ with $P_n\left(x^*\right)$. You may be asking yourself why we would need to approximate $f(x)$ if the function f is known and given.

Example 7.1. *If say $f(x) = \sin(x)$ then why would we need to approximate say $f(1) = \sin(1)$ since any basic calculator could easily compute $\sin(1)$? Well, what your calculator is actually computing is an approximation to $\sin(1)$ using a (rational) number such as $P_n(1)$ and using a large value of n for accuracy (although modern numerical algorithms for computing trigonometric functions have superseded Taylor approximations but Taylor approximations are a good start). Taylor's theorem provides an expression for the remainder term $R_n(x)$ using the derivative $f^{(n+1)}$.*

Theorem 7.1. *(Taylor polynomial with generalized remainder.) Let $f, g : [a, b] \to \mathbb{R}$ be two functions such that for some $n \in \mathbb{N}$ the functions $f, f^{(1)}, f^{(2)}, \ldots, f^{(n)}$ are continuous on $[a, b]$ and $f^{(n+1)}$ exists on (a, b), we suppose that g is continuous on $[a, b]$ and g' exists on (a, b) such $g'(x) \neq 0, \forall x \in (a, b)$. Fix $x_0 \in [a, b]$. Then for any $x \in [a, b]$ there exists c between x_0 and x such that*

$$
f(x) = \sum_{k=0}^{n} \frac{f^{(k)}\left(x_0\right)}{k!}\left(x - x_0\right)^k + R_n(x)
$$

where

$$
R_n(x) = -\frac{f^{(n+1)}(c)\,(x - c)^n\,(g(x) - g(x_0))}{n!\,g'(c)}.
$$

Proof. Suppose that $x_0 < x$, we consider the function $\phi : [x_0, x] \to \mathbb{R}$ defined by

$$
\phi(t) = f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(t)}{k!}(x - t)^k.
$$

ϕ is continuous on $[x_0, x]$ and admits a derivative on (x_0, x) equal to

$$\phi^{'}(t) = -\sum_{k=0}^{n} \frac{f^{(k+1)}(t)(x-t)^k - kf^{(k)}(t)(x-t)^{k-1}}{k!} = -\frac{f^{(n+1)}(t)(x-t)^n}{n!}.$$

Since ϕ and g verify the extended mean value theorem conditions on $[x_0, x]$ then

$$\frac{\phi(x) - \phi(x_0)}{g(x) - g(x_0)} = \frac{\phi'(c)}{g'(c)}, \quad x_0 < c < x.$$

We proceed similarly when $x_0 > x$ by considering $|x - x_0|$ □

Theorem 7.2. *(Taylor polynomial with Lagrange remainder.) Let $f : [a, b] \to \mathbb{R}$ be a function such that for some $n \in \mathbb{N}$ the functions $f, f^{(1)}, f^{(2)}, \ldots, f^{(n)}$ are continuous on $[a, b]$ and $f^{(n+1)}$ exists on (a, b). Fix $x_0 \in [a, b]$. Then for any $x \in [a, b]$ there exists c between x_0 and x such that*

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + R_n(x)$$

where

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}.$$

This is Lagrange form of the remainder.

Proof. Special case of the above theorem by

$$g(t) = (t - x)^{n+1}.$$

and the proof is complete. □

Example 7.2. *Consider the function $f : [0, 2] \to \mathbb{R}$ given by $f(x) = \ln(1 + x)$. Use P_4 based at $x_0 = 0$ to estimate $\ln(2)$ and give a bound on the error with your estimation.*

Solution

Note that $f(1) = \ln(2)$ and so the estimate of $\ln(2)$ using P_4 is $\ln(2) \approx P_4(1)$. To determine P_4 we need $f(0), f^{(1)}(0), \ldots, f^{(4)}(0)$. We compute

$$\begin{aligned}
f^{(1)}(x) &= \frac{1}{1+x} & f^{(1)}(0) &= 1 \\
f^{(2)}(x) &= \frac{-1}{(1+x)^2} & f^{(2)}(0) &= -1 \\
f^{(3)}(x) &= \frac{2}{(1+x)^3} & f^{(3)}(0) &= 2 \\
f^{(4)}(x) &= \frac{-6}{(1+x)^4} & f^{(4)}(0) &= -6.
\end{aligned}$$

Therefore,

$$P_4(x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4.$$

Now $P_4(1) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} = \frac{7}{12}$ and therefore

$$\ln(2) \approx P_4(1) = \frac{7}{12}.$$

The error is $R_4(1) = f(1) - P_4(1)$ which is unknown but we can approximate it using Taylor's theorem. To that end, by Taylor's theorem, for any $x \in [0, 2]$ there exists c in between $x_0 = 0$ and x such that

$$\begin{aligned}
R_4(x) &= \frac{f^{(5)}(c)}{5!}x^5 \\
&= \frac{1}{5!}\frac{24}{(1+c)^5}x^4 \\
&= \frac{1}{5(1+c)^5}.
\end{aligned}$$

Therefore, for $x = 1$, there exists $0 < c < 1$ such that

$$R_4(1) = \frac{1}{5(1+c)^5}.$$

Therefore, a bound for the error is

$$|R_4(1)| = \left|\frac{1}{5(1+c)^5}\right| \leq \frac{1}{5}$$

since $1 + c > 1$.

Theorem 7.3. *(Taylor polynomial with Cauchy remainder.) Let $f : [a, b] \to \mathbb{R}$ be a function such that for some $n \in \mathbb{N}$ the functions $f, f^{(1)}, f^{(2)}, \ldots, f^{(n)}$ are continuous on $[a, b]$ and $f^{(n+1)}$ exists on (a, b). Fix $x_0 \in [a, b]$. Then for any $x \in [a, b]$ there exists c between x_0 and x such that*

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + R_n(x)$$

where

$$R_n(x) = \frac{(x-x_0)^{n+1}(1-\theta)^n}{n!} f^{(n+1)}(x_0+\theta(x-x_0))$$

This is Cauchy form of the remainder.

Proof. Special case of the above theorem by

$$g(t) = (t-x), \quad c = x_0+\theta(x-x_0)).$$

and the proof is complete. □

Corollary 7.1. *Let $n \in \mathbb{N}$ and $\{a_0, a_1, ..., a_n\} \subset \mathbb{R}$. Then for all $x_0 \in \mathbb{R}$, the polynomial* $p(x) = a_0 + a_1 x^2 + ... + a_n x^n$, $x \in \mathbb{R}$. *(1)*
Can be rewritten by the form
$p(x) = b_0 + b_1(x-x_0) + b_2(x-x_0)^2 + ... + b_n(x-x_0)^n, x \in \mathbb{R}$. *(2)*
Where $\{b_0, b_1, ..., b_n\} \subset \mathbb{R}$.

Proof. Set $x = x_0$ in the equation (2), we obtain $b_0 = p(x_0)$. Derivation of (2), we get

$$p'(x) = b_1 + 2b_2(x-x_0) + ... + nb_n(x-x_0)^{n-1}$$

Then by setting $x = x_0, b_1 = p'(x_0)$.
The second derivative, one has $p''(x) = 2!b_2 + ... + n(n-1)b_n(x-x_0)^{n-2}$ and for $x = x_0$ we obtain $p''(x_0) = 2!b_2$, thus $b_2 = \frac{p''(x_0)}{2!}$, In order to determine the others coefficients of (2), we repeat the same technique, we get the general formula

$$b_k = \frac{p^{(k)}(x_0)}{k!} \quad (k = 0, 1, 2, ..., n) \qquad (3)$$

Finally, we obtain Taylor polynomial by introducing coefficients from the equation (3) in the development(2).

$$\begin{aligned} p(x) &= p(x_0) + p'(x_0)(x-x_0) + \frac{p''(x_0)}{2!}(x-x_0)^2 + ... + \frac{p^{(n)}(x_0)}{n!}(x-x_0)^n \\ &= \sum_{k=0}^{n} \frac{p^{(k)}(x_0)}{k!}(x-x_0)^k \quad (4) \end{aligned}$$

□

Example 7.3. *Expand the polynomial $p(x) = x^4 - 5x^3 + 5x^2 + x + 2$ according to the powers of $x-2$.*

We have $x_0 = 2$, *then*

$$p'(x) = 4x^3 - 15x^2 + 10x + 1,\ p''(x) = 12x^2 - 30x + 10,\ p'''(x) = 24x - 30,\ p^{(4)}(x) = 24$$

and

$$p(2) = 0, p'(2) = -7, p''(2) = -2, p'''(2) = 18, p^{(4)}(2) = 24$$

so

$$\begin{aligned} p(x) &= p(2) + \frac{p'(2)}{1!}(x-2) + \frac{p''(2)}{2!}(x-2)^2 + \frac{p'''(2)}{3!}(x-2)^3 + \frac{p^{(4)}(2)}{4!}(x-2)^4 \\ &= -7(x-2) + (-1)(x-2)^2 + 3(x-2)^3 + (x-2)^4 \\ &= -7(x-2) - (x-2)^2 + 3(x-2)^3 + (x-2)^4 \end{aligned}$$

Taylor polynomial with Young remainder or Peano remainder

Theorem 7.4. *Let* $f : [a,b] \to \mathbb{R}$ *be a function such that* $f \in C^n$ *on* $[a,b]$. *Fix* $x_0 \in [a,b]$. *Then there exists a function* ϵ :*[a, b]*$\to \mathbb{R}$ *which satisfies* $\lim_{x\to x_0} \epsilon(x) = 0$, *such that for all* $x \in [a,b]$,

$$f(x) = f(x_0) + \sum_{k=1}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + (x-x_0)^n \epsilon(x).$$

Proof. Let $x_0 \in [a,b]$. We rewrite the Taylor polynomial, with Lagrange remainder to the order $n-1$ on the interval $[x_0, x]$(or $[x, x_0]$); so there exists $c_x \in [x_0, x]$ such that

$$\begin{aligned} f(x) &= f(x_0) + \sum_{k=1}^{n-1} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + \frac{(x-x_0)^n}{n!} f^{(n)}(c_x) \\ &= f(x_0) + \sum_{k=1}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + \frac{(x-x_0)^n}{n!}\left(f^{(n)}(c_x) - f^{(n)}(x_0)\right) \qquad (*). \end{aligned}$$

We set for $x \neq x_0$,

$$\epsilon(x) = \frac{1}{(x-x_0)^n}\left(f(x) - f(x_0) - \sum_{k=1}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k\right)$$

and, since $f^{(n)}$ is continuous at x_0, we deduce from the equality $(*)$ that $\lim_{x\to x_0} \epsilon(x) = 0$.

□

In the next we state a theorem more stronger

Theorem 7.5. *Let $x_0 \in [a,b]$. We suppose that the function $f : [a,b] \to \mathbb{R}$ is n^{th} derivable at point x_0. Then there exists a function $\epsilon : [a,b] \to \mathbb{R}$ which satisfies $\lim_{x\to x_0} \epsilon(x) = 0$ such that for all $x \in [a,b]$,*

$$f(x) = f(x_0) + \sum_{k=1}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + (x-x_0)^n \epsilon(x).$$

Proof. By induction on n. Let $n \in \mathbb{N}$ and set H_n the proposition: for all function $f : [a,b] \to \mathbb{R}, n^{th}$ times derivable at point x_0, we have :

$$\lim_{x\to x_0, x\neq x_0} \frac{1}{(x-x_0)^n}\left(f(x) - f(x_0) - \sum_{k=1}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k\right) = 0$$

H_1 is obviously true, since the function is derivable at point x_0. Suppose that H_n is true and consider the function $f : [a,b] \to \mathbb{R}$, which is $(n+1)^{nt}$ derivable at point x_0. The derivative function f', which is defined on a certain subset $J = [a,b] \cap]\, x_0 - \eta_1, x_0 + \eta_1[$, is n^{th} times derivable at x_0. Let $\epsilon > 0$. There exists $\eta_\epsilon > 0$ such that, for all $t \in [a,b] \cap]\, x_0 - \eta_\epsilon, x_0 + \eta_\epsilon[$, we have :

$$\left| f'(t) - f'(x_0) - \sum_{k=1}^{n} \frac{f^{(k+1)}(x_0)}{k!}(t-x_0)^k \right| \leq \epsilon |t-x_0|^n$$

We define on $[a,b] \cap]x_0 - \eta_\epsilon, x_0 + \eta_\epsilon[$ the following derivable functions h and g by

$$h(t) = f(t) - f(x_0) - \sum_{k=1}^{n+1} \frac{f^{(k)}(x_0)}{k!}(t-x_0)^k$$

and

$$g(t) = \frac{\epsilon}{n+1}|t-x_0|^n (t-x_0)$$

Since H_n is true, implies that

$$\forall t \in [a,b] \cap]x_0 - \eta_\epsilon, x_0 + \eta_\epsilon\,[, \quad |h'(t)| \leq g'(t)$$

Using the mean value theorem, we got

$$\forall x \in [a,b] \cap]x_0 - \eta_\epsilon, x_0 + \eta_\epsilon[, \quad |h(x) - h(x_0)| \leq |g(x) - g(x_0)|$$

which means

$$\forall x \in [a,b] \cap]x_0 - \eta_\epsilon, x_0 + \eta_\epsilon\,[, \quad \frac{1}{|x-x_0|^{n+1}} \left| f(x) - f(x_0) - \sum_{k=1}^{n+1} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k \right| \leq \frac{\epsilon}{n+1} \leq \epsilon.$$

so H_{n+1} is true.

$\square$

Remark 7.1. *In the particular case when $x_0 = 0$, we obtain Maclaurin polynomial with Young (Peano) form of the remainder.*

$$f(x) = f(0) + \frac{f'(0)}{1!}x + ... + \frac{f^n(0)}{n!}x^n + x^n\epsilon(x).$$

Example 7.4. $f(x) = \sin x$. *We have*

$$\sin^{(n)}(x) = \sin\left(x + n\frac{\pi}{2}\right), \quad then \quad f^{(n)}(0) = \sin n\frac{\pi}{2} = \begin{cases} 0, & n = 2k \\ (-1)^k, & n = 2k+1 \end{cases} \quad (k \in \mathbb{N})$$

Thus Maclaurin polynomial with Young (Peano) form of the remainder, takes the following form

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + ... + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + x^{2n+2}\epsilon(x), x \longrightarrow 0.$$

Remark 7.2. *Lagrange form of the remainder can take the form*

$$R_n(x) = \frac{f^{(n+1)}(x_0 + \theta(x - x_0))}{(n+1)!}(x - x_0)^{n+1}.$$

Example 7.5. *Let $n \in \mathbb{N}$, $\forall x \in \mathbb{R}$, $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + ... + \frac{x^n}{n!} + R_n(x)$, where*

$$R_n(x) = \frac{f^{(n+1)}(\theta x)}{(n+1)!}x^{n+1} = \frac{e^{\theta x}}{(n+1)!}x^{n+1}, \theta \in \left]0,1\right[.$$

Example 7.6. *Let $n \in \mathbb{N}, \forall x \in \mathbb{R}$, $\sin x = x + \frac{x^3}{3!} + ... + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + R_{2n+1}(x)$, where*

$$R_{2n+1}(x) = (-1)^{n+1} \frac{\sin \theta x}{(2n+2)!}x^{2n+2}, \theta \in \left]0,1\right[.$$

Example 7.7. *Let $n \in \mathbb{N}$, $\forall x \in \mathbb{R}$, $\cos x = 1 + \frac{x^2}{2!} + ... + (-1)^n \frac{x^{2n}}{(2n)!} + R_{2n}(x)$, where*

$$R_{2n}(x) = (-1)^{n+1} \frac{\sin \theta x}{(2n+1)!}x^{2n+1}, \theta \in \left]0,1\right[.$$

Example 7.8. *Let $n \in \mathbb{N}, \forall x > -1$, $\ln(1+x) = x - \frac{x^2}{2} + ... + (-1)^{n-1}\frac{x^n}{n} + R_n(x)$,* *where*

$$R_n(x) = (-1)^n \frac{1}{(n+1)(1+\theta x)^{n+1}} x^{n+1}, \theta \in \left]0,1\right[.$$

Example 7.9. *Let $n \in \mathbb{N}$, for all $x > -1$,*
$(1+x)^\alpha = 1 + \frac{\alpha}{1!}x + \frac{\alpha(\alpha-1)}{2!}x^2 + ... + \frac{\alpha(\alpha-1)...\alpha(\alpha-n+1)}{n!}x^n + R_n(x)$,
where

$$R_n(x) = \frac{\alpha(\alpha-1)...\alpha(\alpha-n)}{(n+1)!}(1+\theta x)^{\alpha-n-1} x^{n-1}, \theta \in \left]0,1\right[.$$

Taylor polynomial with integral reminder

Theorem 7.6. *Let $f : [a,b] \to \mathbb{R}$ be a function such that $f \in C^{(n+1)}([a,b])$. Fix $x_0 \in [a,b]$. Then for any $x \in [a,b]$ we have*

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + \frac{1}{n!}\int_{x_0}^{x}(x-t)^n f^{(n+1)}(t)dt$$

Proof. By induction technique on n, and by using integration by parts on $\frac{1}{n!}\int_{x_0}^{x}(x-t)^n f^{n+1}(t)dt$, we conclude. □

7.2 Big O and Little o Notation-Bachmann–Landau notation

It is often useful to talk about the rate at which some function changes as its argument grows (or shrinks), without worrying to much about the detailed form. This is what the O(·) and o(·) notation lets us do. Let x_0 be an accumulation point of a subset D, $f : D \to \mathbb{R}$, $g : D \to \mathbb{R}$.

Definition 7.2. *We say that f is negligible compared to g or is ultimately smaller than, when $x \to x_0$ and we note $f = o(g)$(little o) if :*

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x : |x - x_0| < \delta \Rightarrow |f(x)| < \varepsilon |g(x)|$$

Corollary 7.2. *It results from definition (7.2) that if g does not vanish on x_0 neighbourhood's then:*

$f = o(g)(x \to x_0) \Leftrightarrow \lim_{x\to x_0} \frac{f(x)}{g(x)} = 0$

If $g = 1$, then

$f = o(1)(x \to x_0) \Leftrightarrow \lim_{x\to x_0} f(x) = 0$

Definition 7.3. *Let f and g be two functions defined on the interval $[a, +\infty[$. we set by definition $f = \circ(g)(x \to +\infty) \Leftrightarrow \forall\varepsilon > 0, \exists A > 0, \forall x : x > A \Rightarrow |f(x)| \leqslant \varepsilon |g(x)|$*

Definition 7.4. *Let x_0 be an accumulation point of a subset D, $f : D \to \mathbb{R}$, $g : D \to \mathbb{R}$. f is said to be equivalent to g, when $x \to x_0$ and we note by $f \sim g$ if $f - g = \circ(1)$, $x \to x_0$.*

Remark 7.3. *If $\forall x \in D/\{x_0\}$: $g(x) \neq 0$. then $f \sim g$, $x \to x_0 \Longleftrightarrow \lim\limits_{x\to x_0} \dfrac{f(x)}{g(x)} = 1$.*

Example 7.10. $\lim\limits_{x\to 0} \dfrac{a^x - 1}{x \ln a} = \lim\limits_{y\to 0} \left(\dfrac{e^y - 1}{y}\right) = 1$, $y = x \ln a \Longleftrightarrow a^x - 1 = x \ln a + \circ(x)$, $x \to 0 \Longleftrightarrow a^x - 1 \sim x \ln a$, $x \to 0$.

Theorem 7.7. *Let $x \in D/\{x_0\} : g(x) \neq 0$, $g_1(x) \neq 0$ and $g \sim g_1$, $x \to x_0$. Then for all function $f : D \to \mathbb{R}$, one has*

$$\lim_{x\to x_0} (f(x)\, g(x)) = \lim_{x\to x_0} (f(x)\, g_1(x))$$

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f(x)}{g_1(x)}.$$

Definition 7.5. *Let x_0 be an accumulation point of a subset D, $f : D \to \mathbb{R}$, $g : D \to \mathbb{R}$. We say that, g is "of the same order" as f, and they "grow at the same rate", or "shrink at the same rate", we say also f is dominated by g when $x \to x_0$ and we write $f = O(g)$ if*

$\exists M > 0, \exists\delta > 0, \forall x : |x - x_0| < \delta \Rightarrow |f(x)| \leqslant M |g(x)|$.

If $x_0 = +\infty$, then f is dominated by g when $x \to +\infty$ if $\exists M > 0, \exists A > 0, \forall x : x > A \Rightarrow |f(x)| \leqslant M |g(x)|$.

Remark 7.4. *IF $\lim\limits_{x\to x_0} \dfrac{f(x)}{g(x)} \in \mathbb{R}$ (exists) then $f = O(g)$ $(x \to x_0)$.*

Remark 7.5. *Reminder that, the notation $f = O(1)$ on D means that f is bounded on D.*

7.3 Limited development in a neighborhood of a point

Let I be an open interval and $f : I \to \mathbb{R}$ be a given function.

Definition 7.6. *For $x_0 \in \overline{I}$ and $n \in \mathbb{N}$, we say that f admits a limited development (expansion) at point x_0 and of order n, if there exist real numbers $c_0, c_1, \ldots, c_n$ and a function $\epsilon : I \to \mathbb{R}$ such that $\lim_{x\to x_0} \epsilon(x) = 0$ and so that for all $x \in I$:*

$$f(x) = c_0 + c_1(x - x_0) + \cdots + c_n(x - x_0)^n + (x - x_0)^n \epsilon(x).$$

- *$c_0 + c_1(x-x_0) + \cdots + c_n(x-x_0)^n$ is called the regular or polynomial part of the limited development.*

- *$(x-x_0)^n\epsilon(x)$ is called the remainder of the limited development.*

Taylor-Young polynomial allows us to express quickly the limited development by setting $c_k = \frac{f^{(k)}(x_0)}{k!}$:

Proposition 7.1. *Suppose that f belongs to C^n on a neighborhood of x_0, then f admits a limited development at the point x_0 of order n.*

$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + (x-x_0)^n\epsilon(x)$ *where* $\lim_{x\to x_0}\epsilon(x) = 0$.

Remark 7.6. *1. The next expression represents the limited development for a function f which belongs to C^n on a neighborhood of 0*

$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + \cdots + f^{(n)}(0)\frac{x^n}{n!} + x^n\epsilon(x)$$

2. If f admits a limited development at a point x_0 of order n so, it admits a limited development for all

$k \leqslant n$. Indeed

$$\begin{aligned} f(x) &= f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \cdots + \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k \\ &+ \underbrace{\frac{f^{(k+1)}(x_0)}{(k+1)!}(x-x_0)^{k+1} + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + (x-x_0)^n\epsilon(x)}_{=(x-x_0)^k\eta(x)} \end{aligned}$$

where $\lim_{x\to x_0}\eta(x) = 0$.

Proposition 7.2. *If f admits a limited development, so, it is unique.*

Proof. By contrapositive, we suppose that it admits two $f : f(x) = c_0 + c_1(x-x_0) + \cdots + c_n(x-x_0)^n + (x-x_0)^n\epsilon_1(x)$ and $f(x) = d_0 + d_1(x-x_0) + \cdots + d_n(x-x_0)^n + (x-x_0)^n\epsilon_2(x)$. We consider the difference, we obtain

$$(d_0 - c_0) + (d_1 - c_1)(x-x_0) + \cdots + (d_n - c_n)(x-x_0)^n + (x-x_0)^n(\epsilon_2(x) - \epsilon_1(x)) = 0.$$

Replacing $x = x_0$, we get $d_0 - c_0 = 0$. Then dividing this equality by $x - x_0$ (or can also proceed by derivation) we obtain $(d_1 - c_1) + (d_2 - c_2)(x-x_0) + \cdots + (d_n - c_n)(x-x_0)^{n-1} + (x-x_0)^{n-1}(\epsilon_2(x) - \epsilon_1(x)) = 0$. Setting $x = x_0$ we deduce $d_1 - c_1 = 0$, etc. So on we show that $c_0 = d_0, c_1 = d_1, \ldots, c_n = d_n$. The polynomials parts are equal, so the remainder. □

Corollary 7.3. *If f is even (resp. odd) then the polynomial part of the limited development is of the same parity (which means that the limited development contains only the even monomials resp. odd ones)).*

Remark 7.7. *1. Suppose that we have the limited development of a given function and if the function belongs to C^n. Then we can calculate the $f^{(k)}(a)$ from the relation $c_k = \frac{f^{(k)}(a)}{k!}$. This is due to the uniqueness of the LD and to Taylor-Young polynomial. Worthy speaking , in the real life we do the opposite.*

2. *If f admits a LD at a point a to order $n \geqslant 0$ then f is continuous at a and $c_0 = f(a)$.*

3. *If f admits un LD at a point a to order $n \geqslant 1$, then f est derivable at a and we have $c_0 = f(a)$ and $c_1 = f'(a)$. Therefore $y = c_0 + c_1(x-a)$ is the tangent equation of f at a.*

4. *Contrary to expectation : f can admit LD to order 2 at a point a without admitting a second derivative at a. For example, let $f(x) = x^3 \sin \frac{1}{x}$. So f is derivable but f' is not. Despite that, f admits a LD at 0 to order $2 : f(x) = x^2\epsilon(x)$ (the polynomial part is equal to zero).*

Limited development of elementary (usual) funtion at origine

Taylor-Young polynomial is a very powerful tool to provide LD for smooth functions, the following table, can testify.

$$\begin{aligned}
\exp x = e^x &= 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + x^n\epsilon(x) \\
\operatorname{ch} x = \cosh x &= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + \frac{x^{2n}}{(2n)!} + x^{2n+1}\epsilon(x) \\
\operatorname{sh} x = \sinh x &= \frac{x}{1!} + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{x^{2n+1}}{(2n+1)!} + x^{2n+2}\epsilon(x) \\
\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + x^{2n+1}\epsilon(x) \\
\sin x &= \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + x^{2n+2}\epsilon(x) \\
\ln(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1}\frac{x^n}{n} + x^n\epsilon(x)
\end{aligned}$$

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \cdots + \frac{\alpha(\alpha-1)\ldots(\alpha-n+1)}{n!}x^n + x^n\epsilon(x)$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots + (-1)^n x^n + x^n \epsilon(x)$$
$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + x^n \epsilon(x)$$
$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{1}{8}x^2 + \cdots + (-1)^{n-1}\frac{1 \cdot 1 \cdot 3 \cdot 5 \cdots (2n-3)}{2^n n!}x^n + x^n \epsilon(x)$$

7.3.1 Limited development of functions at any given point

The function f admits un LD on a neighborhood of a point a if and only if the function $x \mapsto f(x+a)$ admits a LD on a neighborhood of 0 . Often we transform the problem back to 0 by making the change of variables $h = x - a$.

Example 7.11. *1. Give the LD of $f(x) = \exp x$ at 1 .*

We set $h = x - 1$. Since x is close to 1 thus h is close to 0. So we are concerned by a LD of $\exp h$ at $h = 0$.

$$\begin{aligned}\exp x &= \exp(1 + (x-1)) = \exp(1)\exp(x-1) = e\exp h \\ &= e\left(1 + h + \frac{h^2}{2!} + \cdots + \frac{h^n}{n!} + h^n \epsilon(h)\right) \\ &= e\left(1 + (x-1) + \frac{(x-1)^2}{2!} + \cdots + \frac{(x-1)^n}{n!} + (x-1)^n \epsilon(x-1)\right)\end{aligned}$$

where $\lim_{x\to 1}\epsilon(x-1) = 0$.

2. Express the LD of $g(x) = \sin x$ at $\pi/2$.

We have $\sin x = \sin\left(\frac{\pi}{2} + x - \frac{\pi}{2}\right) = \cos\left(x - \frac{\pi}{2}\right)$ so we are face to LD of $\cos h$, when $h = x - \frac{\pi}{2} \to 0$. Thus: $\sin x = 1 - \frac{\left(x-\frac{\pi}{2}\right)^2}{2!} + \cdots + (-1)^n \frac{\left(x-\frac{\pi}{2}\right)^{2n}}{(2n)!} + \left(x - \frac{\pi}{2}\right)^{2n+1}\epsilon\left(x - \frac{\pi}{2}\right)$, où $\lim_{x\to\pi/2}\epsilon\left(x - \frac{\pi}{2}\right) = 0$.

3. Express the LD of $f(x) = \ln(1+3x)$ at 1 to order 3.

We put $h = x - 1$, so $x = 1 + h$. One has $f(x) = \ln(1+3x) = \ln(1+3(1+h)) = \ln(4+3h) = \ln\left(4\cdot\left(1+\frac{3h}{4}\right)\right) = \ln 4 + \ln\left(1+\frac{3h}{4}\right) = \ln 4 + \frac{3h}{4} - \frac{1}{2}\left(\frac{3h}{4}\right)^2 + \frac{1}{3}\left(\frac{3h}{4}\right)^3 + h^3\epsilon(h) = \ln 4 + \frac{3(x-1)}{4} - \frac{9}{32}(x-1)^2 + \frac{9}{64}(x-1)^3 + (x-1)^3\epsilon(x-1)$ where $\lim_{x\to 1}\epsilon(x-1) = 0$.

7.3.2 Operations on limited developments

Sum and product

Let f and g be two functions which have limited development at 0 of order n :

$f(x) = c_0 + c_1 x + \cdots + c_n x^n + x^n \epsilon_1(x) \quad g(x) = d_0 + d_1 x + \cdots + d_n x^n + x^n \epsilon_2(x)$

Proposition 7.3.

- *$f+g$ admits LD at 0 to order n defined by $(f+g)(x)=f(x)+g(x)=(c_0+d_0)+(c_1+d_1)\,x+\cdots+(c_n+d_n)\,x^n+x^n\epsilon(x)$.*
- *$f\times g$ admits a LD at 0 to order n as $(f\times g)(x)=f(x)\times g(x)=T_n(x)+x^n\epsilon(x)$ where $T_n(x)$ is the polynomial $(c_0+c_1x+\cdots+c_nx^n)\times(d_0+d_1x+\cdots+d_nx^n)$ truncated to order n.*

Example 7.12. *Calculate the LD of $\cos x\times\sqrt{1+x}$ at 0 to order 2.*
From previous table, we have $\cos x=1-\frac{1}{2}x^2+x^2\epsilon_1(x)$ and $\sqrt{1+x}=1+\frac{1}{2}x-\frac{1}{8}x^2+x^2\epsilon_2(x)$. Therefore

$$
\begin{aligned}
&\cos x\times\sqrt{1+x}=\left(1-\frac{1}{2}x^2+x^2\epsilon_1(x)\right)\times\left(1+\frac{1}{2}x-\frac{1}{8}x^2+x^2\epsilon_2(x)\right)\\
&=1+\frac{1}{2}x-\frac{1}{8}x^2+x^2\epsilon_2(x)-\frac{1}{2}x^2\left(1+\frac{1}{2}x-\frac{1}{8}x^2+x^2\epsilon_2(x)\right)\\
&+x^2\epsilon_1(x)\left(1+\frac{1}{2}x-\frac{1}{8}x^2+x^2\epsilon_2(x)\right)\\
&=1+\frac{1}{2}x-\frac{1}{8}x^2+x^2\epsilon_2(x)-\frac{1}{2}x^2-\frac{1}{4}x^3+\frac{1}{16}x^4-\frac{1}{2}x^4\epsilon_2(x)\\
&+x^2\epsilon_1(x)+\frac{1}{2}x^3\epsilon_1(x)-\frac{1}{8}x^4\epsilon_1(x)+x^4\epsilon_1(x)\epsilon_2(x)\\
&=\underbrace{1+\frac{1}{2}x+\left(-\frac{1}{8}x^2-\frac{1}{2}x^2\right)}\ \textit{polynomial of order } 2\\
&+\underbrace{x^2\epsilon_2(x)-\tfrac{1}{4}x^3+\tfrac{1}{16}x^4-\tfrac{1}{2}x^4\epsilon_2(x)+x^2\epsilon_1(x)+\tfrac{1}{2}x^3\epsilon_1(x)-\tfrac{1}{8}x^4\epsilon_1(x)+x^4\epsilon_1(x)\epsilon_2(x)}_{\textit{Remainder of the form } x^2\epsilon(x)}\\
&=1+\frac{1}{2}x-\frac{5}{8}x^2+x^2\epsilon(x)
\end{aligned}
$$

Composition

Let be

$$
\begin{aligned}
f(x)&=C(x)+x^n\epsilon_1(x)=c_0+c_1x+\cdots+c_nx^n+x^n\epsilon_1(x)\\
g(x)&=D(x)+x^n\epsilon_2(x)=d_0+d_1x+\cdots+d_nx^n+x^n\epsilon_2(x)
\end{aligned}
$$

Proposition 7.4. *If $g(0)=0$ (i.e $d_0=0$) then the function $f\circ g$ admits a LD at 0 of order n, so that the regular part is the truncated polynomial from the composition $C(D(x))$ of order n.*

Example 7.13. *Calculate the LD of $h(x)=\sin(\ln(1+x))$ at 0 of order 3.*

One has $f\circ g(x)=\sin(\ln(1+x))$ and $g(0)=0$. Since the LD of order 3 according to $f(u)=\sin u$ for u close to 0 is $f(u)=\sin u=u-\frac{u^3}{3!}+u^3\epsilon_1(u)$. Set $u=g(x)=\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}+x^3\epsilon_2(x)$ for x near 0. Then $h(x)=f\circ g(x)=f(u)=u-\frac{u^3}{3!}+u^3\epsilon_1(u)=\left(x-\frac{1}{2}x^2+\frac{1}{3}x^3\right)-\frac{1}{6}x^3+x^3\epsilon(x)=x-\frac{1}{2}x^2+\frac{1}{6}x^3+x^3\epsilon(x)$.

Example 7.14. *Let $h(x) = \sqrt{\cos x}$. The LD of h at 0 to order 4 is*

The LD of $f(u) = \sqrt{1+u}$ at $u = 0$ of order 2 is of the form $f(u) = \sqrt{1+u} = 1 + \frac{1}{2}u - \frac{1}{8}u^2 + u^2\epsilon(u)$. Let us put $u(x) = \cos x - 1$, thus $h(x) = f(u(x))$ and $u(0) = 0$. From other hand, the Ld of $u(x)$ at $x = 0$ to order 4 is $u = -\frac{1}{2}x^2 + \frac{1}{24}x^4 + x^4\epsilon(x)$. Therefore $u^2 = \frac{1}{4}x^4 + x^4\epsilon(x)$, and so

$$\begin{aligned}
h(x) = f(u) &= 1 + \frac{1}{2}u - \frac{1}{8}u^2 + u^2\epsilon(u) \\
&= 1 + \frac{1}{2}\left(-\frac{1}{2}x^2 + \frac{1}{24}x^4\right) - \frac{1}{8}\left(\frac{1}{4}x^4\right) + u^4\epsilon(u) \\
&= 1 - \frac{1}{4}x^2 + \frac{1}{48}x^4 - \frac{1}{32}x^4 + u^4\epsilon(u) \\
&= 1 - \frac{1}{4}x^2 - \frac{1}{96}x^4 + u^4\epsilon(u)
\end{aligned}$$

Division

To calculate the LD of a quotient f/g, we proceed as follow. Let $f(x) = c_0 + c_1x + \cdots + c_nx^n + x^n\epsilon_1(x) \quad g(x) = d_0 + d_1x + \cdots + d_nx^n + x^n\epsilon_2(x)$ be two limited developments of suitable functions. We consider the LD of $\frac{1}{1+u} = 1 - u + u^2 - u^3 + \cdots$. Then we have two cases:

1. If $d_0 = 1$, we set $u = d_1x + \cdots + d_nx^n + x^n\epsilon_2(x)$ and the quotient takes the form $f/g = f \times \frac{1}{1+u}$.
2. If d_0 is any non null real, then we return back to the first case by considering $\frac{1}{g(x)} = \frac{1}{d_0}\frac{1}{1+\frac{d_1}{d_0}x+\cdots+\frac{d_n}{d_0}x^n+\frac{x^n\epsilon_2(x)}{d_0}}$.
3. If $d_0 = 0$, then we factorize by a suitable x^k (smartly choose k) to come back to previous cases.

Example 7.15. *1. Determine the LD of $\tan x$ at 0 of order 5.*

First of all, we recall that $\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} + x^5\epsilon(x)$ and $\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} + x^5\epsilon(x) = 1 + u$ where $u = -\frac{x^2}{2} + \frac{x^4}{24} + x^5\epsilon(x)$. Then

$$\begin{aligned}
\frac{1}{\cos x} = \frac{1}{1+u} &= 1 - u + u^2 - u^3 + u^3\epsilon(u) \\
&= 1 + \frac{x^2}{2} - \frac{x^4}{24} + \frac{x^4}{4} + x^5\epsilon(x) \\
&= 1 + \frac{x^2}{2} + \frac{5}{24}x^4 + x^5\epsilon(x);
\end{aligned}$$

Therefore

$$\begin{aligned}
\tan x &= \sin x \times \frac{1}{\cos x} \\
&= \left(x - \frac{x^3}{6} + \frac{x^5}{120} + x^5\epsilon(x)\right) \times \left(1 + \frac{x^2}{2} + \frac{5}{24}x^4 + x^5\epsilon(x)\right) \\
&= x + \frac{x^3}{3} + \frac{2}{15}x^5 + x^5\epsilon(x).
\end{aligned}$$

2. *The LD of $\frac{1+x}{2+x}$ at 0 of order 4 is obtained by the following way*

$$\begin{aligned}
\frac{1+x}{2+x} &= (1+x)\frac{1}{2}\frac{1}{1+\frac{x}{2}} \\
&= \frac{1}{2}(1+x)\left(1 - \frac{x}{2} + \left(\frac{x}{2}\right)^2 - \left(\frac{x}{2}\right)^3 + \left(\frac{x}{2}\right)^4 + o\left(x^4\right)\right) \\
&= \frac{1}{2} + \frac{x}{4} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{x^4}{32} + o\left(x^4\right)
\end{aligned}$$

3. *Calculate the LD of $\frac{\sin x}{\operatorname{sh} x}$ at 0 of order 4. Indeed we proceed as follow*

$$\begin{aligned}
\frac{\sin x}{\operatorname{sh} x} &= \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} + o\left(x^5\right)}{x + \frac{x^3}{3!} + \frac{x^5}{5!} + o\left(x^5\right)} = \frac{x\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} + o\left(x^4\right)\right)}{x\left(1 + \frac{x^2}{3!} + \frac{x^4}{5!} + o\left(x^4\right)\right)} \\
&= \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} + o\left(x^4\right)\right) \times \frac{1}{1 + \frac{x^2}{3!} + \frac{x^4}{5!} + o\left(x^4\right)} \\
&= \cdots = 1 - \frac{x^2}{2} + \frac{x^4}{18} + o\left(x^4\right)
\end{aligned}$$

Example 7.16. *The limited development of $\frac{2+x+2x^3}{1+x^2}$ to order 2 can be calculated by setting $C(x) = 2+x+2x^3$ and $g(x) = D(x) = 1+x^2$ thus $C(x) = D(x)\times(2+x-2x^2)+x^3(1+2x)$. Therefore we have $Q(x) = 2+x-2x^2$, $R(x) = 1+2x$. So by diving this equality by $D(x)$ we obtain $\frac{f(x)}{g(x)} = 2 + x - 2x^2 + x^2\epsilon(x)$.*

Integration

let $f : I \to \mathbb{R}$ be a function of class C^n so that the LD at $a \in I$ to order n is given by $f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + (x-a)^n\epsilon(x)$.

Theorem 7.8. *Note F a primitive of f. Thus F admits a LD at $a \in I$ of order $n+1$ which can be written as*

$$\begin{aligned}
&F(x) = F(a) + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \cdots \\
&\cdots + c_n\frac{(x-a)^{n+1}}{n+1} + (x-a)^{n+1}\eta(x) \\
&\textit{where } \lim_{x\to a}\eta(x) = 0.
\end{aligned}$$

Proof. We have $F(x)-F(a) = \int_a^x f(t)dt = a_0(x-a)+\cdots+\frac{a_n}{n+1}(x-a)^{n+1}+\int_a^x(t-a)^n\epsilon(t)dt$. Put $\eta(x) = \frac{1}{(x-a)^{n+1}}\int_a^x(t-a)^n\epsilon(t)dt$. (Remark : ϵ is continuous, indeed, it is by definition continuous at a, also it is continuous outside of a since
$\epsilon(x) = \frac{1}{(x-a)^n}\left(f(x) - (c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + \ c_n(x-a)^n)\ \right).)$
Then $|\eta(x)| \leqslant \left|\frac{1}{(x-a)^{n+1}}\int_a^x\right|(t-a)^n\left|\cdot \sup_{t\in[a,x]}\right|\epsilon(t)|dt| = \left|\frac{1}{(x-a)^{n+1}}\right|\cdot\sup_{t\in[a,x]}|\epsilon(t)|\cdot\int_a^x|(t-a)^n|\,dt = \frac{1}{n+1}\sup_{t\in[a,x]}|\epsilon(t)|$.

But $\sup_{t\in[a,x]}|\epsilon(t)| \to 0$ when $x\to a$. So $\eta(x)\to 0$ while $x\to a$.

$\square$

Example 7.17. *Calculate LD of* $\arctan x$ *at 0.*

From previous results, we know that $\arctan' x = \frac{1}{1+x^2}$. *Put* $f(x) = \frac{1}{1+x^2}$ *and* $F(x) = \arctan x$*, we can have*

$$\arctan' x = \frac{1}{1+x^2} = \sum_{k=0}^{n}(-1)^k x^{2k} + x^{2n}\epsilon(x).$$

Since $\arctan(0) = 0$ *then* $\arctan x = \sum_{k=0}^{n}\frac{(-1)^k}{2k+1}x^{2k+1}+x^{2n+1}\epsilon(x) = x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots$

The LD of arcsin x at 0 of order 5 is given by:

$\arcsin' x = (1-x^2)^{-\frac{1}{2}} = 1 - \frac{1}{2}(-x^2) + \frac{-\frac{1}{2}\left(-\frac{1}{2}-1\right)}{2}(-x^2)^2 + x^4\epsilon(x) = 1 + \frac{1}{2}x^2 + \frac{3}{8}x^4 + x^4\epsilon(x)$.

Thus $\arcsin x = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + x^5\epsilon(x)$.

7.3.3 Extended (generalized) limited development, extended limited development at infinity

Let $f : \mathcal{D}_f \subseteq \mathbb{R} \to \mathbb{R}$ be a function.

1. We say that f admits an extended limited development at $x_0 \in \mathbb{R}$ to order $n \in \mathbb{N}$ if

$$f(x) = \frac{b_p}{(x-x_0)^p}+\frac{b_{p-1}}{(x-x_0)^{p-1}}+\cdots+\frac{b_1}{x-x_0}+a_0+a_1(x-x_0)+\cdots+a_n(x-x_0)^n+o\left((x-x_0)^n\right)$$

In the neighborhood of x_0, we can define right generalized limited development and left generalized limited development.

2. If $\mathcal{D}_f$ contains an interval of the form $]a,+\infty[$, we say that f admits an extended limited development at $+\infty$ of order $n \in \mathbb{N}$ if

$$f(x) = b_p x^p + b_{p-1}x^{p-1} + \cdots + b_1 x + a_0 + \frac{a_1}{x} + \frac{a_2}{x^2} + \cdots + \frac{a_n}{x^n} + o\left(\frac{1}{x^n}\right)$$

Same definition on neighborhood of $-\infty$.

Some applications of Limited developments

In what follows we give some important applications of limited developments

Limit calculations

Limited developments are one of the most powerful tools used to face indeterminate forms. If $f(x) = c_0 + c_1(x-a) + \cdots$ then $\lim_{x\to a} f(x) = c_0$.

Example 7.18. *Find the limit at 0 of* $\frac{\ln(1+x)-\tan x+\frac{1}{2}\sin^2 x}{3x^2\sin^2 x}$.
Let us name this fraction by $\frac{f(x)}{g(x)}$.
The limited developments at 0 are given by

$$f(x) = \ln(1+x) - \tan x + \tfrac{1}{2}\sin^2 x = \left(x - \tfrac{x^2}{2} + \tfrac{x^3}{3} - \tfrac{x^4}{4} + o\left(x^4\right)\right) - \left(x + \tfrac{x^3}{3} + o\left(x^4\right)\right) + \tfrac{1}{2}\left(x - \tfrac{x^3}{6} + o\left(x^3\right)\right)^2 = -\tfrac{x^2}{2} - \tfrac{x^4}{4} + \tfrac{1}{2}\left(x^2 - \tfrac{1}{3}x^4\right) + o\left(x^4\right) = -\tfrac{5}{12}x^4 + o\left(x^4\right)$$

and

$$g(x) = 3x^2\sin^2 x = 3x^2(x+o(x))^2 = 3x^4 + o\left(x^4\right).$$

Thus $\frac{f(x)}{g(x)} = \frac{-\frac{5}{12}x^4+o\left(x^4\right)}{3x^4+o(x^4)}$. *So* $\lim_{x\to 0}\frac{f(x)}{g(x)} = -\frac{5}{36}$.

7.3.4 Position of a curve relative to its tangent

Proposition 7.5. *Let* $f: I \to \mathbb{R}$ *be a function such that its LD at a is given by* $f(x) = c_0 + c_1(x-a) + c_k(x-a)^k + (x-a)^k\epsilon(x)$, *where* $k \geq 2$ *is the smallest natural number such that the coefficient* c_k *does not vanish. Then the equation of the tangent of the curve of f at a is* $y = c_0 + c_1(x-a)$. *The position of the curve relative to the tangent for x close to a is determinated by the sign of* $f(x) - y$, *in other words by* $c_k(x-a)^k$.

Three possible cases:

1. *If this sign is positive then the curve is above the tangent equation.*
2. *If this sign is negative then the curve is below the tangent equation.*
3. *If this sign changes (when going from $x < a$ to $x > a$) then the curve crosses the tangent equation at the abscissa point a. It is an inflection point.*

As the DL of f in a to order 2 is also written $f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + (x-a)^2\epsilon(x)$, *then the tangent equation is also* $y = f(a) + f'(a)(x-a)$. *If in addition* $f''(a) \neq 0$ *then* $f(x) - y$ *keeps a constant sign around a. Consequently if a is an inflection point then* $f''(a) = 0$. *(The converse is false.)*

Example 7.19. *Let $f(x) = x^4 - 2x^3 + 1$.*

1. *Determine the tangent equation at $\frac{1}{2}$ of the curve of f and specify the position of the curve relative to the tangent equation.*

We have $f'(x) = 4x^3 - 6x^2, f''(x) = 12x^2 - 12x$, so $f''\left(\frac{1}{2}\right) = -3 \neq 0$ et $k = 2$. We deduce the LD of f at $\frac{1}{2}$ by Taylor-Young formula
$f(x) = f\left(\frac{1}{2}\right) + f'\left(\frac{1}{2}\right)\left(x - \frac{1}{2}\right) + \frac{f''\left(\frac{1}{2}\right)}{2!}\left(x - \frac{1}{2}\right)^2 + \left(x - \frac{1}{2}\right)^2 \epsilon(x) = \frac{13}{16} - \left(x - \frac{1}{2}\right) - \frac{3}{2}(x - \frac{1}{2})^2 + \left(x - \frac{1}{2}\right)^2 \epsilon(x)$

So the tangent equation at $\frac{1}{2}$ is $y = \frac{13}{16} - \left(x - \frac{1}{2}\right)$ and the curve of f is below the tangent equation since $f(x) - y = \left(-\frac{3}{2} + \epsilon(x)\right)\left(x - \frac{1}{2}\right)^2$ *is negative round the point $x = \frac{1}{2}$.*

2. *Inflection points*

The inflection points are to be sought among the solutions of $f''(x) = 0$. So among $x = 0$ and $x = 1$.

- *The LD at 0 is $f(x) = 1 - 2x^3 + x^4$. Thus $y = 1$ is the tangent equation at the point of abscissa 0. Since the sign of $-2x^3$ changes, therefore 0 is an inflection point of f.*

- *The LD at* 1 *is given by $f(x) = -2(x-1) + 2(x-1)^3 + (x-1)^4$. Then $y = -2(x-1)$ is the tangent equation at the point of abscissa 1. Since $2(x-1)^3$ changes its sign at 1, so 1 is an inflection point too of f.*

7.3.5 Limited development at $+\infty$

Let f be a function defined on an interval $I =]x_0, +\infty[$. We say that f admits a LD at $+\infty$ of order n if there exist real numbers $c_0, c_1, \ldots, c_n$ such that

$$f(x) = c_0 + \frac{c_1}{x} + \cdots + \frac{c_n}{x^n} + \frac{1}{x^n}\epsilon\left(\frac{1}{x}\right)$$

where $\epsilon\left(\frac{1}{x}\right)$ goes to 0 when $x \to +\infty$.

Example 7.20.

$f(x) = \ln\left(2 + \frac{1}{x}\right) = \ln 2 + \ln\left(1 + \frac{1}{2x}\right) = \ln 2 + \frac{1}{2x} - \frac{1}{8x^2} + \frac{1}{24x^3} + \cdots + (-1)^{n-1}\frac{1}{n2^n x^n} + \frac{1}{x^n}\epsilon\left(\frac{1}{x}\right)$, *où* $\lim_{x\to\infty} \epsilon\left(\frac{1}{x}\right) = 0$

This allows us to have a fairly precise idea of the behavior of f in the neighborhood of $+\infty$. When $x \to +\infty$ then $f(x) \to \ln 2$, and the second term is $+\frac{1}{2x}$, so is positive, it means that the function $f(x)$ tends to $\ln 2$ *while remaining above* $\ln 2$.

Remark 7.8. *1. A LD at $+\infty$ is said to be called asymptotic development too.*

2. A function $x \mapsto f(x)$ admits a LD in $+\infty$ of order n is equivalent to saying that the function $x \to f\left(\frac{1}{x}\right)$ admits a LD at 0^+ of order n.

3. We can define in the same way what is a LD at $-\infty$.

Proposition 7.6. *We assume that the function $x \mapsto \frac{f(x)}{x}$ admits a DL in $+\infty$ (or in $-\infty$): $\frac{f(x)}{x} = a_0 + \frac{a_1}{x} + \frac{a_k}{x^k} + \frac{1}{x^k}\epsilon\left(\frac{1}{x}\right)$, where k is the smallest integer $\geqslant 2$ such that the coefficient of $\frac{1}{x^k}$ is non-zero. Then $\lim_{x\to+\infty} f(x) - (a_0x + a_1) = 0$ (resp. $x \to -\infty$): the line $y = a_0x + a_1$ is an asymptote to the curve of f at $+\infty$(or $-\infty$) and the position of the curve with respect to the asymptote is given by the sign of $f(x) - y$, i.e. the sign of $\frac{a_k}{x^{k-1}}$.*

Proof. We have $\lim_{x\to+\infty} (f(x) - a_0x - a_1) = \lim_{x\to+\infty} \frac{a_k}{x^{k-1}} + \frac{1}{x^{k-1}}\epsilon\left(\frac{1}{x}\right) = 0$. So $y = a_0x + a_1$ is an asymptote to the curve of f. Then we calculate the difference $f(x) - a_0x - a_1 = \frac{a_k}{x^{k-1}} + \frac{1}{x^{k-1}}\epsilon\left(\frac{1}{x}\right) = \frac{a_k}{x^{k-1}}\left(1 + \frac{1}{a_k}\epsilon\left(\frac{1}{x}\right)\right)$. □

Example 7.21. *Determine the asymptotes of $f(x) = \exp\frac{1}{x} \cdot \sqrt{x^2-1}$, if any.*

1. At $+\infty$,

$$\begin{aligned}
\frac{f(x)}{x} &= \exp\frac{1}{x} \cdot \frac{\sqrt{x^2-1}}{x} = \exp\frac{1}{x} \cdot \sqrt{1 - \frac{1}{x^2}} \\
&= \left(1 + \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3} + \frac{1}{x^3}\epsilon\left(\frac{1}{x}\right)\right) \cdot \left(1 - \frac{1}{2x^2} + \frac{1}{x^3}\epsilon\left(\frac{1}{x}\right)\right) \\
&= \cdots = 1 + \frac{1}{x} - \frac{1}{3x^3} + \frac{1}{x^3}\epsilon\left(\frac{1}{x}\right)
\end{aligned}$$

So the asymptote of f at $+\infty$ is $y = x+1$. Since $f(x) - x - 1 = -\frac{1}{3x^2} + \frac{1}{x^2}\epsilon\left(\frac{1}{x}\right)$ when $x \to +\infty$, the curve of f remains below the asymptote.

2. At $-\infty$. $\frac{f(x)}{x} = \exp\frac{1}{x} \cdot \frac{\sqrt{x^2-1}}{x} = -\exp\frac{1}{x} \cdot \sqrt{1 - \frac{1}{x^2}} = -1 - \frac{1}{x} + \frac{1}{3x^3} + \frac{1}{x^3}\epsilon\left(\frac{1}{x}\right)$. So $y = -x - 1$ is an asymptote of f at $-\infty$. We have $f(x) + x + 1 = \frac{1}{3x^2} + \frac{1}{x^2}\epsilon\left(\frac{1}{x}\right)$ when $x \to -\infty$; The curve of f remains above the asymptote.

CHAPTER

8

INDEFINITE INTEGRALS

Introduction

In this chapter, our main goal is to introduce the basic concepts in integral calculus where we present the different techniques of integration which will be useful tool to carry on for example with definite integrals and the differential equations.

8.1 indefinite integrals

Definition 8.1. *Let f be a function of a closed interval $[a,b]$ in $\mathbb{R}$ and let F be a differentiable function on $[a,b]$. F is said to be primitive of f on $[a,b]$ if*

$$\forall x \in [a,b], F'(x) = f(x).$$

Proposition 8.1. *If F and G are two primitives of f on $[a,b]$, then*

$$F - G = c, c \in \mathbb{R}.$$

Proof. Indeed, we have $(F-G)'(x) = F'(x) - G'(x) = 0, \forall x \in [a,b]$ then $F-G$ is a constant function on $[a,b]$. □

Example 8.1. *The functions F and G defined on $[1,2]$ by $F(x) = \ln x$ and $G(x) = \ln x + \alpha$, with $\alpha \in \mathbb{R}$ are two primitives of the function $f(x) = \frac{1}{x}$ on $[1,2]$.*

Definition 8.2. *The set of all primitives of the function $f : [a,b] \to \mathbb{R}$ is called the indefinite integral of f, denoted $\int f(x)dx$, then if F is a primitive of f on $[a,b]$, we have*

$$\int f(x)dx = F(x) + c, c \in \mathbb{R}.$$

Example 8.2. $\forall x \in [1,2] : \int \frac{1}{x}dx = \ln x + c, c \in \mathbb{R}$

Remark 8.1. *A function f admitting a primitive on $[a,b]$, is not necessarily continuous on $[a,b]$.*

Example 8.3. *Let be the function f defined by*

$$f(x) = \begin{cases} 2x\sin\frac{1}{x} - \cos\frac{1}{x}, & si\ x \in]0,1] \\ 0, & si\ x = 0 \end{cases}$$

f admits as a primitive on $[0,1]$ the function F defined by

$$F(x) = \begin{cases} x^2\sin\frac{1}{x}, & si\ x \in]0,1] \\ 0, & si\ x = 0 \end{cases}, \text{ since } F'(x) = f(x), \forall x \in [0,1].$$

while f is discontinuous at $x = 0$.

Proposition 8.2. *Let f and g be two functions which admit two primitives on $[a,b]$, then $f+g$ and αf where $\alpha \in \mathbb{R}$ admit primitives and we have*

1. $\int (f+g)(x)dx = \int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx.$
2. $\int (\alpha f)(x)dx = \int \alpha f(x)dx = \alpha \int f(x)dx.$
3. $(\int f(x)dx)' = f(x).$
4. $\int f'(x)dx = f(x) + c, c \in \mathbb{R}.$

Proof. Just do it. □

Proposition 8.3. *Let $[a,b]$ be an interval, and $f : [a,b] \longmapsto \mathbb{R}$ a function defined on the interval $[a,b]$. If f is a continuous function on the interval $[a,b]$, then f admits a primitive F defined for all $x \in [a,b]$ by $F(x) = \int_a^x f(t)dt$.*

In this case, F is the unique primitive of f which vanishes at a. This result is known as the Fundamental Theorem of calculus. But this is not the true fundamental theorem of calculus (this is a corollary), see Riemann integral. Thus, it suffices that a function be continuous on an interval for it to admit a primitive on this one.

Primitive functions of elementary functions

$$\int a dx = ax + c, \quad \int x dx = \frac{x^2}{2} + c, \quad \int x^m dx = \frac{x^{m+1}}{m+1} + c \quad m \in N$$
$$\int \frac{dx}{x^2} = -\frac{1}{x} + c \quad x \neq 0, \quad \int \frac{dx}{2\sqrt{x}} = \sqrt{x} + c \quad x \neq 0$$
$$\int x^\alpha dx = \frac{x^{\alpha+1}}{\alpha+1} + c \quad \alpha \in \mathbb{Q} - (-1), \quad \int \frac{dx}{x} = \ln|x| + c \quad x \neq 0$$
$$\int e^x dx = e^x + c, if \quad a > 0 \quad \text{and } a \neq 1, \quad \int a^x dx = \frac{a^x}{\ln a} + c \qquad x \neq k\pi \quad k \in Z$$
$$\int \sin x dx = -\cos x + c, \quad \int \cos x dx = \sin x + c$$
$$\int \frac{dx}{\cos^2 x} = \int \left(1 + \tan^2 x\right) dx = \tan x + c \quad k \in Z, \quad \int \frac{dx}{\sin^2 x} = -\operatorname{cotan} x + c$$
$$\int \operatorname{sh} x dx = \operatorname{ch} x + c, \quad \int \operatorname{ch} x dx = \operatorname{sh} x + c$$
$$\int \operatorname{th} x dx = \ln(\operatorname{ch} x) + c, \quad \int \frac{dx}{\operatorname{ch}^2 x} = \int \left(1 - \operatorname{th}^2 x\right) dx = \operatorname{th} x + c$$
$$\int \cos^2 x dx = \frac{x}{2} + \frac{\sin 2x}{4} + c, \quad \int \sin^2 x dx = \frac{x}{2} - \frac{\sin 2x}{4} + c \quad x \neq \frac{\pi}{2} + k\pi \text{ and } k \in Z$$
$$\int \tan x dx = -\ln|\cos x| + c, \quad \int \ln x dx = x\ln x - x + c, \quad x > 0$$

$$\int \frac{dx}{\sin x} = \ln\left|\tan\left(\frac{x}{2}\right)\right| + c \quad x \neq (2k+1)\pi \quad and \quad k \quad \in Z$$
$$\int \frac{dx}{\cos x} = \ln\left|\tan\left(\frac{x}{2} + \frac{\pi}{4}\right)\right| + c \quad x \neq \frac{\pi}{2} + 2k\pi \quad and \quad k \quad \in Z$$
$$\int \frac{dx}{1-x^2} = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right| + c \quad |x| \neq 1 \quad (\quad or \quad argthx \quad if \quad x \in]-1,1[$$

$\int \frac{dx}{\sqrt{1-x^2}} = \operatorname{Arcsin x} = -\operatorname{Arccos x} + c \quad |x| < 1$
$\int \frac{dx}{1+x^2} = \arctan x + c$
$\int \frac{dx}{\sqrt{x^2+1}} = \ln\left|x + \sqrt{x^2+1}\right| + c \quad$ (or argshx)
$\int \frac{dx}{\sqrt{x^2-1}} = \ln\left|x + \sqrt{x^2-1}\right| + c \quad |x| > 1 \quad$ (or argchx if $x > 1$)
$\int \frac{dx}{\sqrt{x^2+h}} = \ln\left|x + \sqrt{x^2+h}\right| + c \quad x^2 + h > 0$
$\int \frac{dx}{\operatorname{sh} x} = \ln\left|\operatorname{th}\left(\frac{x}{2}\right)\right| + c \quad x > 0 \quad$ or $\quad x < 0$
$\int \frac{dx}{\operatorname{ch} x} = 2\operatorname{Arctan}\left(e^x\right) + c$
$\int \frac{dx}{\operatorname{th} x} = \ln|\operatorname{sh} x| + c \quad x > 0 \quad$ or $\quad x < 0$

8.2 Methods for computing primitive functions

8.2.1 Integration by parts -IBP- or Integration per partes - IPP -

Theorem 8.1. *Let u and v be two derivable functions of class C^1 on $[a, b]$; then :*

$$\int u'(x)v(x)dx = u(x)v(x) - \int u(x)v'(x)dx$$

Proof. Indeed

$$u(x)v(x) = \int [u(x)v(x)]'dx = \int u'(x)v(x)dx + \int u(x)v'(x)dx.$$

□

Remark 8.2. 1. *Sometimes, we have to apply IPP method more than one's time to solve the problem.*

2. *We can also use differentials of functions as follow*

$$\int udv = uv - \int vdu$$

such that $df = f'(x)dx$

Examples 8.1. 1. $I_1 = \int xe^x dx$

IPP : $\begin{cases} u = x \\ dv = e^x dx \end{cases} \Rightarrow \begin{cases} du = dx \\ v = e^x \end{cases}$

thus $I_1 = xe^x - \int e^x dx = e^x(x-1) + c, c \in \mathbb{R}$.

2. $I_2 = \int \arctan x dx$

IPP : $\begin{cases} u = \arctan x \\ dv = dx \end{cases} \Rightarrow \begin{cases} du = \frac{1}{1+x^2}dx \\ v = x \end{cases}$

so $I_2 = x \arctan x - \int \frac{x}{1+x^2}dx = x \arctan x - \frac{1}{2}\ln(1+x^2) + c, c \in \mathbb{R}$.

3. $I_3 = \int (\ln x)^2 dx$

IPP 1 : $\begin{cases} u = (\ln x)^2 \\ dv = dx \end{cases} \Rightarrow \begin{cases} du = 2\frac{\ln x}{x}dx \\ v = x \end{cases}$

therefore $I_3 = x(\ln x)^2 - 2\int \ln x dx = x(\ln x)^2 - 2J$

IPP 2 : $\begin{cases} u = \ln x \\ dv = dx \end{cases} \Rightarrow \begin{cases} du = \frac{1}{x}dx \\ v = x \end{cases}$

then $J = x \ln x - \int dx = x(\ln x - 1) + c, c \in \mathbb{R}$.

so $I_3 = x(\ln x)^2 - 2x(\ln x - 1) + c', \quad c' \in \mathbb{R}$.

4. $I_4 = \int e^x \cos x dx$

$$IPP\ 1: \begin{cases} u = e^x \\ dv = \cos x dx \end{cases} \Rightarrow \begin{cases} du = e^x dx \\ v = \sin x \end{cases}$$
so $I_4 = e^x \sin x - \int e^x \sin x dx = e^x \sin x - J$
$$\text{IPP 2}: \begin{cases} u = e^x \\ dv = \sin x dx \end{cases} \Rightarrow \begin{cases} du = e^x dx \\ v = -\cos x \end{cases}$$
we get $J = -e^x \cos x + \int e^x \cos x dx = -e^x \cos x + I_4 + c, c \in \mathbb{R}$.

then $I_4 = e^x \sin x + e^x \cos x - I_4 - c \Leftrightarrow I_4 = \frac{e^x}{2}[\sin x + \cos x] + c', c' \in \mathbb{R}$.

Example 8.4. $I = \int (x^2 - 1) e^x dx$
$$IPP \begin{cases} u = x^2 - 1 \\ dv = e^x dx \end{cases} \Rightarrow \begin{cases} du = 2x dx \\ v = e^x \end{cases}$$

$$so\ I = \left(x^2 - 1\right) e^x - 2 \int x e^x dx$$

$$\begin{cases} u = x \\ dv = e^x dx \end{cases} \Rightarrow \begin{cases} du = dx \\ v = e^x \end{cases}$$
$$then \int x e^x dx = x e^x - \int e^x dx = e^x (x - 1) + c, c \in \mathbb{R}$$

thus $I = (x^2 - 1) e^x - 2e^x (x - 1) + c', c' \in \mathbb{R}$.

8.2.2 Change of variables - CV -

In mathematics, a change of variables is a basic technique used to simplify problems in which the original variables are replaced with functions of other variables. The intent is that when expressed in new variables, the problem may become simpler, or equivalent to a better understood problem.

Change of variables is an operation that is related to substitution. However these are different operations, as can be seen when considering differentiation (chain rule) or integration (integration by substitution).

Theorem 8.2. *Let* $f : [a, b] \to \mathbb{R}$ *be a continuous function on* $[a, b]$ *and let* $\varphi : [\alpha, \beta] \to [a, b]$ *be a derivable function of class* C^1 *on* $[a, b]$*, such that* $\varphi([\alpha, \beta]) \subset [a, b]$ *then*

$$\int f(x) dx = \int f(\varphi(t)) \varphi'(t) dt.$$

By the change of variables, we conclude.

$$x = \varphi(t) \Rightarrow dx = \varphi'(t) dt.$$

The following table gives us a summary of some well known cases

If $a \neq 0$, then one has:

$\int f'(ax+b)dx = \frac{1}{a}f(ax+b)+c$

$\int (ax+b)^\alpha dx = \frac{1}{a}\frac{(ax+b)^{\alpha+1}}{\alpha+1}+c \quad \alpha \neq -1 \quad (ax+b \neq 0 \text{ si } \alpha < 0)$

$\int \frac{dx}{ax+b} = \frac{1}{a}\ln|ax+b|+c \quad ax+b \neq 0$

more general cases

$\int f'[u(x)]u'(x)dx = f[u(x)]+c$

$\int [u(x)]^\alpha u'(x)dx = \frac{[u(x)]^{\alpha+1}}{\alpha+1}+c \quad \alpha \neq -1 \quad (u(x) \neq 0 \text{ si } \alpha < 0)$

$\int \frac{u'(x)}{u(x)}dx = \ln|u(x)|+c \quad u(x) \neq 0$

$\int_a^b f(x)dx = \int_{t_a}^{t_b} f[\phi(t)]\phi'(t)dt$ with ϕ monotone and differentiable on $[t_a, t_b]$ and $a = \phi(t_a), b = \phi(t_b)$

Example 8.5. *1.*

$I_1 = \int \cos x \cdot e^{\sin x}dx$

$CV : t = \sin x \Rightarrow dt = \cos x dx$, *we get*

$I_1 = \int e^t dt = e^t + c, c \in \mathbb{R}$, *so* $I_1 = e^{\sin x} + c, c \in \mathbb{R}$.

2. $I_2 = \int \frac{e^x}{e^{2x}+1}dx$

$CV : t = e^x \Rightarrow dt = e^x dx$, *thus*

$I_2 = \int \frac{dt}{t^2+1} = \arctan t + c, c \in \mathbb{R}$, *therefore* $I_2 = \arctan(e^x) + c, c \in \mathbb{R}$.

3. $I_3 = \int \frac{\arcsin^3 x}{\sqrt{1-x^2}}dx$

$CV : t = \arcsin x \Rightarrow dt = \frac{1}{\sqrt{1-x^2}}dx$, *so*

$I_3 = \int t^3 dt = \frac{1}{4}t^4 + c, c \in \mathbb{R}$ *then* $I_3 = \frac{1}{4}(\arcsin x)^4 + c, c \in \mathbb{R}$.

Integrals involving quadratic expressions

Calculate $I_1 = \int \frac{dx}{ax^2+bx+c}$

Step 1: We transform the denominator by putting it in the canonical form i.e, the sum or the difference of two squares

$$ax^2+bx+c = a\left[\left(x+\frac{b}{2a}\right)^2 + \frac{4ac-b^2}{4a^2}\right],$$

Let $\frac{4ac-b^2}{4a^2} = \pm M^2$, then

$$ax^2+bx+c = \begin{cases} a\left[\left(x+\frac{b}{2a}\right)^2 - M^2\right], & \text{if } \Delta > 0 \\ a\left[\left(x+\frac{b}{2a}\right)^2 + M^2\right], & \text{if } \Delta < 0. \end{cases}$$

Thus I_1 takes the following form

$$I_1 = \int \frac{dx}{a\left[\left(x+\frac{b}{2a}\right)^2 \pm M^2\right]} = \frac{1}{aM^2}\int \frac{dx}{\left[\left(\frac{2ax+b}{2aM}\right)^2 \pm 1\right]}$$

Step 2: By the change of variables

$$t = \frac{2ax+b}{2aM} \Rightarrow dt = \frac{dx}{M} \Leftrightarrow dx = Mdt$$

we get

$$I_1 = \frac{1}{aM}\int \frac{dt}{t^2 \pm 1}$$

so :

$\underline{1}^{st}$ Case : If $I_1 = \frac{1}{aM}\int \frac{dt}{t^2+1}$ then

$$I_1 = \frac{1}{aM}\arctan t + c, c \in \mathbb{R}$$

therefore

$$I_1 = \frac{1}{aM}\arctan\left(\frac{2ax+b}{2aM}\right) + c, c \in \mathbb{R}$$

$\underline{2}^{\text{sd}}$ Case : If $I_1 = \frac{1}{aM}\int \frac{dt}{t^2-1}$ then

$$I_1 = \frac{1}{2aM}\ln\left|\frac{t-1}{t+1}\right| + c, c \in \mathbb{R}$$

so

$$I_1 = \frac{1}{aM}\ln\left|\frac{2ax+b-2aM}{2ax+b+2aM}\right| + c, c \in \mathbb{R}$$

Examples 8.2. *1.* $I = \int \frac{dx}{x^2+2x+5}$

One has

$$x^2+2x+5 = (x+1)^2 - 1 + 5 = (x+1)^2 + 4,$$

so

$$I = \int \frac{dx}{x^2+2x+5} = \int \frac{dx}{(x+1)^2+4} = \frac{1}{4}\int \frac{dx}{\left(\frac{x+1}{2}\right)^2+1}$$

The change of variable

$$t = \frac{x+1}{2} \Rightarrow dt = \frac{1}{2}dx \Leftrightarrow dx = 2dt,$$

permits us to calculate

$$I = \frac{1}{4}\int \frac{2dt}{t^2+1} = \frac{1}{2}\int \frac{dt}{t^2+1} = \frac{1}{2}\arctan t + c, \quad c \in \mathbb{R}.$$

therefore

$$I = \frac{1}{2}\arctan\left(\frac{x+1}{2}\right) + c, \quad c \in \mathbb{R}.$$

2. *Calculate* $I_2 = \int \frac{\alpha x+\beta}{ax^2+bx+c}dx$

   Step1 : Derivation of the denominator $(ax^2+bx+c)' = 2ax+b$.

   Step2: We rewrite the numerator according to the derivative of the denominator.

$$\alpha x + \beta = \alpha\left(x + \frac{\beta}{\alpha}\right) = \frac{\alpha}{2a}\left(2ax + \frac{2a\beta}{\alpha} + b - b\right)$$

so

$$\alpha x + \beta = \frac{\alpha}{2a}\left[(2ax+b) + \frac{2a\beta}{\alpha} - b\right]$$

then we replace in I_2

$$\begin{aligned} I_2 &= \frac{\alpha}{2a}\int \frac{(2ax+b) + \left(\frac{2a\beta}{\alpha} - b\right)}{ax^2+bx+c}dx \\ &= \frac{\alpha}{2a}\int \frac{(2ax+b)}{ax^2+bx+c}dx + \left(\beta - \frac{b\alpha}{2a}\right)\int \frac{dx}{ax^2+bx+c}, \end{aligned}$$

hen

$$I_2 = \frac{\alpha}{2a}\ln\left|ax^2+bx+c\right| + \left(\beta - \frac{b\alpha}{2a}\right) I_1$$

where I_1 *has been calculated above.*

Example 8.6. $I = \int \frac{(3x-1)}{x^2-x+1}dx$.
Using the above technique, we have

$$I = \frac{3}{2}\int \frac{(2x-1)+\frac{1}{3}}{x^2-x+1}dx = \frac{3}{2}\int \frac{(2x-1)}{x^2-x+1}dx + \frac{1}{2}\int \frac{dx}{x^2-x+1}$$

so

$$I = \frac{3}{2}\ln\left(x^2-x+1\right) + \frac{1}{2}J$$

where

$$J = \int \frac{dx}{x^2-x+1}$$

thus

$$J = \frac{2}{\sqrt{3}}\arctan\left(\frac{2x-1}{\sqrt{3}}\right) + c, \quad c \in \mathbb{R}.$$

and
$$I=\frac{3}{2}\ln\left(x^2-x+1\right)+\frac{1}{\sqrt{3}}\arctan\left(\frac{2x-1}{\sqrt{3}}\right)+c,\quad c\in\mathbb{R}$$

3. *Calculate* $I_3=\int\frac{dx}{\sqrt{ax^2+bx+c}}$

   We transform the ax^2+bx+c *by putting it in the canonical form i.e, the sum or the difference of two squares, then we proceed by the same change of variables,*

$$\text{CV}: t=\frac{2ax+b}{2aM}\Rightarrow dt=\frac{dx}{M}\Leftrightarrow dx=Mdt$$

   to obtain one the two following possible integrals

$$\begin{cases} I_3=\int\frac{dt}{\sqrt{t^2\pm1}}, \ \textit{where } t^2-1>0 \ \textit{if } a>0\\ I_3=\int\frac{dt}{\sqrt{1-t^2}}, \ \textit{where } 1-t^2>0 \ \textit{if } a<0\end{cases}$$

   Case number 1 : *If* $I_3=\int\frac{dt}{\sqrt{t^2+1}}$*, then*

$$I_3=\arg sht+c, c\in\mathbb{R}.$$

   Case number2 : If $I_3=\int\frac{dt}{\sqrt{t^2-1}}$ *and* $t^2-1>0$*, then*

$$I_3=\arg cht+c, c\in\mathbb{R}.$$

   Case number 3 : *:If* $I_3=\int\frac{dt}{\sqrt{1-t^2}}$ *and* $1-t^2>0$*, then*

$$I_3=\arcsin t+c,\quad c\in\mathbb{R}.$$

Example 8.7. *Calculate a primitive of* $I=\int\frac{dx}{\sqrt{x^2+x+1}}$*. Let*

$$x^2+x+1=\left(x+\frac{1}{2}\right)^2-\frac{1}{4}+1=\left(x+\frac{1}{2}\right)^2+\frac{3}{4}$$

we obtain

$$I=\int\frac{dx}{\sqrt{x^2+x+1}}=\int\frac{dx}{\sqrt{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}}=\frac{2}{\sqrt{3}}\int\frac{dx}{\sqrt{\left(\frac{2x+1}{\sqrt{3}}\right)^2+1}}$$
$$CV: t=\frac{2x+1}{\sqrt{3}}\Rightarrow dt=\frac{2}{\sqrt{3}}dx\Leftrightarrow dx=\frac{\sqrt{3}}{2}dt$$

so

$$I = \int \frac{dt}{\sqrt{t^2+1}} = \arg sht + c, c \in \mathbb{R},$$
$$I = \arg sh\left(\frac{2x+1}{\sqrt{3}}\right) + c, c \in \mathbb{R}.$$

4. *Calculate* $I_4 = \int \frac{\alpha x+\beta}{\sqrt{ax^2+bx+c}}dx$

   Step 1: We derivate $(ax^2+bx+c)' = 2ax+b$.

   Step 2: We rewrite the numerator with respect to the derivative, as did while calculating I_2,

$$\alpha x + \beta = \frac{\alpha}{2a}\left[(2ax+b) + \frac{2a\beta}{\alpha} - b\right]$$

then we replace in I_4

$$\begin{aligned} I_4 &= \frac{\alpha}{2a}\int \frac{(2ax+b) + \left(\frac{2a\beta}{a} - b\right)}{\sqrt{ax^2+b+c}}dx \\ &= \frac{\alpha}{2a}\int \frac{(2ax+b)}{\sqrt{ax^2+bx+c}}dx + \left(\beta - \frac{b\alpha}{2a}\right)\int \frac{dx}{\sqrt{ax^2+bx+c}} \\ &= \frac{\alpha}{2a}\int \frac{(2ax+b)}{\sqrt{ax^2+bx+c}}dx + \left(\beta - \frac{ba}{2a}\right) I_3 \end{aligned}$$

for the first primitive, the following change of variable suffices

$$t = ax^2 + bx + c \Rightarrow dt = (2ax+b)dx$$

thus

$$\int \frac{(2ax+b)}{2\sqrt{ax^2+bx+c}}dx = \int \frac{dt}{2\sqrt{t}} = \sqrt{t} + k, k \in \mathbb{R}.$$
$$= \sqrt{ax^2+bx+c} + k, k \in \mathbb{R}.$$

so

$$I_4 = \frac{\alpha}{a}\sqrt{ax^2+bx+c} + \left(\beta - \frac{b\alpha}{2a}\right) I_3$$

where I_3 *is above.*

Example 8.8. *Give a primitive of* $I = \int \frac{5x+3}{\sqrt{10+4x+x^2}}dx$. *We remark, that*

$$\left(10+4x+x^2\right)' = 2x+4$$
$$5x+3 = 5\left(x+\frac{3}{5}\right) = \frac{5}{2}\left(2x+\frac{6}{5}+4-4\right)$$
$$= \frac{5}{2}\left[(2x+4) - \frac{14}{5}\right]$$

so

$$I=\frac{5}{2}\int\frac{(2x+4)-\frac{14}{5}}{\sqrt{10+4x+x^2}}dx=\frac{5}{2}\int\frac{(2x+4)}{\sqrt{10+4x+x^2}}dx-7\int\frac{dx}{\sqrt{10+4x+x^2}}$$

for the first integral, we consider

$$t=10+4x+x^2\Rightarrow dt=(2x+4)dx$$

then

$$\begin{aligned}\int\frac{(2x+4)}{2\sqrt{10+4x+x^2}}dx=&\int\frac{dt}{2\sqrt{t}}=\sqrt{t}+c_1,c_1\in\mathbb{R},\\ =&\sqrt{10+4x+x^2}+c_1,c_1\in\mathbb{R}\end{aligned}$$

for the second integral, we factorize $10+4x+x^2$

$$10+4x+x^2=(x+2)^2+6$$

than we replace in the integral

$$\int\frac{dx}{\sqrt{10+4x+x^2}}=\int\frac{dx}{\sqrt{(x+2)^2+6}}=\frac{1}{\sqrt{6}}\int\frac{dx}{\sqrt{\left(\frac{x+2}{\sqrt{6}}\right)^2+1}}$$

the change of variable $t=\frac{x+2}{\sqrt{6}}\Rightarrow dt=\frac{1}{\sqrt{6}}dx\Leftrightarrow dx=\sqrt{6}dt$
gives

$$\int\frac{dx}{\sqrt{10+4x+x^2}}=\int\frac{dt}{\sqrt{t^2+1}}=\operatorname{arg\,sh}t+c_2,c_2\in\mathbb{R}$$

therefore

$$I=5\sqrt{10+4x+x^2}-7\operatorname{arg\,sh}\left(\frac{x+2}{\sqrt{6}}\right)+c,\ \textit{where } c=c_1+c_2.$$

8.2.3 Integration of rational functions

Definition 8.3. *Recall that a rational function is a ratio of two polynomials P and Q i.e. $f(x)=\frac{P(x)}{Q(x)}$, which is well defined for all x of $\mathbb{R}$ such that $Q(x)\neq 0$.*

In order to integrate $f(x)=\frac{P(x)}{Q(x)}$, we should distinguish two cases
1[st] Case : if $d^\circ P\geq d^\circ Q$ (where d° is the order), then we establish an euclidean division relative to decreasing powers of x, indeed

$$P(x)=Q(x)\cdot S(x)+R(x),$$

where $S(x)$ and $R(x)$ are two polynomials such that $d^\circ R<d^\circ Q$, then

$$f(x)=\frac{P(x)}{Q(x)}=S(x)+\frac{R(x)}{Q(x)}$$

2^{sd} case: if $d°P < d°Q$, to integrate a proper rational function, we can apply the method of partial fractions. This method allows to turn the integral of a complicated rational function into the sum of integrals of simpler functions. The denominators of the partial fractions can contain nonrepeated linear factors, repeated linear factors, nonrepeated irreducible quadratic factors, and repeated irreducible quadratic factors.

To evaluate integrals of partial fractions, we use the following formulas:

Decomposition of rational functions to partial fractions

In order to integrate a rational function, it is reduced to a proper rational function. The method in which the integrand is expressed as the sum of simpler rational functions is known as decomposition into partial fractions. We decompose the function f according to the denominator. Steps for Decomposing a Rational Function of the Appropriate Type Into a Sum of Ratios

Step 1: Factor the denominator if necessary.

Step 2: Create a sum of rational terms for each factor, using different variables for each numerator.

Step 3: Multiply by the LCD to clear the fractions.

Step 4: Solve for the undetermined coefficients.

Step 5: Substitute the solved coefficients into the sum of ratios. Worthy speaking, we proceed as follow

- If

$$Q(x)=(x-a_1)(x-a_2)\ldots(x-a_n),$$

where $a_i\in\mathbb{R},\forall i=1,..,n$, then

$$f(x)=\frac{P(x)}{Q(x)}=\frac{A_1}{x-a_1}+\frac{A_2}{x-a_2}+\ldots+\frac{A_n}{x-a_n}.$$

such that A_i, $i=1,..,n$ are real constants to find.

- If

$$Q(x)=(x-a_1)^{m_1}(x-a_2)^{m_2}\ldots(x-a_n)^{m_n},$$

where $a_i \in \mathbb{R}, m_i \in \mathbb{N}^*, \forall i = 1, .., n$, then

$$\frac{P(x)}{Q(x)} = \left[\frac{A_1^1}{x-a_1} + \frac{A_1^2}{(x-a_1)^2} + .. + \frac{A_1^{m_1}}{(x-a_1)^{m_1}}\right] + \left[\frac{A_2^1}{x-a_2} + \frac{A_2^2}{(x-a_2)^2} + .. + \frac{A_2^{m_2}}{(x-a_2)^{m_2}}\right] + \ldots + \left[\frac{A_n^2}{x-a_n} + \frac{A_n^2}{(x-a_n)^2} + .. + \frac{A_n^2}{(x-a_n)^{m_n}}\right].$$

such that A_i^j, $i = 1, \ldots, n$ and $j = 1, \ldots, m_i$ are real constants to determine.

- If

$$Q(x) = \left(x^2 + b_1 x + c_1\right)\left(x^2 + b_2 x + c_2\right)\ldots\left(x^2 + b_n x + c_n\right),$$

where $b_i, c_i \in \mathbb{R}$ and $b_i^2 - 4c_i < 0, \forall i = 1, \ldots, n$ then

$$\frac{P(x)}{Q(x)} = \frac{A_1 x + B_1}{x^2 + b_1 x + c_1} + \frac{A_2 x + B_2}{x^2 + b_2 x + c_2} + .. + \frac{A_n x + B_n}{x^2 + b_n x + c_n}$$

such that A_i, B_i are real constants to determine, for all $i = 1, \ldots, n$. - If

$$Q(x) = \left(x^2 + b_1 x + c_1\right)^{m_1}\left(x^2 + b_2 x + c_2\right)^{m_2}\ldots\left(x^2 + b_n x + c_n\right)^{m_n}$$

where $b_i, c_i \in \mathbb{R}$ and $b_i^2 - 4c_i < 0, m_i \in \mathbb{N}^*, \forall i = 1, .., n$, then

$$\begin{aligned}\frac{P(x)}{Q(x)} =& \left[\frac{A_1^1 x + B_1^1}{x^2 + b_1 x + c_1} + \frac{A_1^2 x + B_1^2}{(x^2 + b_1 x + c_1)^2} + .. + \frac{A_1^{m_1} x + B_1^{m_1}}{(x^2 + b_1 x + c_1)^{m_1}}\right] \\ &+ \left[\frac{A_2^1 x + B_2^1}{x^2 + b_2 x + c_2} + \frac{A_2^2 x + B_2^2}{(x^2 + b_2 x + c_2)^2} + .. + \frac{A_2^{m_2} x + B_2^{m_2}}{(x^2 + b_2 x + c_2)^{m_2}}\right] + \\ &\ldots + \left[\frac{A_n^1 x + B_n^1}{x^2 + b_n x + c_n} + \frac{A_n^2 x + B_n^2}{(x^2 + b_n x + c_n)^2} + .. + \frac{A_n^{m_n} x + B_n^{m_n}}{(x^2 + b_n x + c_n)^{m_n}}\right].\end{aligned}$$

such that A_i^j are real constant to determine for all $i = 1, \ldots, n$ and $j = 1, .., m_i$.

Examples 8.3. *Decompose the following functions into a sum of ratios.*

1. $f(x) = \frac{x+3}{x^2-3x+2}$.

We have, $x^2 - 3x + 2 = (x-1)(x-2)$, *then*

$$f(x) = \frac{x+3}{(x-2)(x-1)} = \frac{A}{x-2} + \frac{B}{x-1}$$

basic calculations give

$$\frac{x+3}{(x-2)(x-1)} = \frac{5}{x-2} - \frac{4}{x-1}.$$

2. $f(x) = \frac{5-x}{(x^2-4x+4)(x+1)}$.

Remark that $x^2 - 4x + 4 = (x-2)^2$, *so we have*

$$f(x) = \frac{5-x}{(x-2)^2(x+1)} = \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{x+1}$$

identification method for example allows us to rewrite it under the form

$$\frac{5-x}{(x-2)^2(x+1)} = \frac{x^2(A+C) + x(-A+B-4C) - 2A + B + 4C}{(x-2)^2(x+1)}$$

resolving the system below

$$\begin{cases} A + C = 0 \\ -A + B - 4C = -1 \\ -2A + B + 4C = 5 \end{cases}$$

gives us

$$\frac{5-x}{(x-2)^2(x+1)} = \frac{-2}{3(x-2)} + \frac{1}{(x-2)^2} + \frac{2}{3(x+1)}.$$

3. $f(x) = \frac{x^2}{(x^2+x+1)(x-1)}$.

The discriminant Δ *of* $x^2 + x + 1$ *is negative, then*

$$f(x) = \frac{x^2}{(x^2+x+1)(x-1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}$$

by identification method, we get

$$\frac{x^2}{(x^2+x+1)(x-1)} = \frac{x^2(A+B) + x(A-B+C) + A - C}{(x^2+x+1)(x-1)}$$

then by solving the system below

$$\begin{cases} A + B = 1 \\ A - B + C = 0 \\ A - C = 0 \end{cases}$$

we get

$$\frac{x^2}{(x^2+x+1)(x-1)} = \frac{1}{3(x-1)} + \frac{2x+1}{3(x^2+x+1)}.$$

4. $f(x) = \frac{4x^3-5}{(x^2-3x+5)(-x^2+x-2)}$.

The discriminant of (x^2-3x+5) *and* $(-x^2+x-2)$ *are negatives, so we have*

$$f(x)=\frac{4x^3-5}{(x^2-3x+5)(-x^2+x-2)}=\frac{Ax+B}{x^2-3x+5}+\frac{Cx+D}{-x^2+x-2},$$

therefore

$$\frac{4x^3-5}{(x^2-3x+5)(-x^2+x-1)}=\frac{\frac{3}{2}x+\frac{3}{2}}{x^2-x+1}-\frac{\frac{11}{2}x+\frac{5}{2}}{x^2-3x+5}$$

5. $f(x)=\frac{2x^5-1}{(2x^2-x+1)(x^2+1)^2}$

$$f(x)=\frac{2x^5-1}{(2x^2-x+1)\left(x^2+1\right)^2}=\frac{Ax+B}{2x^2-x+1}+\frac{Cx+D}{x^2+1}+\frac{Ex+F}{\left(x^2+1\right)^2}$$

therefore

$$\frac{2x^5-1}{(2x^2-x+1)(x^2+1)^2}=\frac{\frac{3}{2}x-\frac{5}{4}}{2x^2-x+1}-\frac{\frac{3}{2}x+\frac{1}{2}}{\left(x^2+1\right)^2}+\frac{\frac{1}{4}x+\frac{3}{4}}{x^2+1}$$

Integration rational functions

For the integration of the rational functions $f(x)=\frac{P(x)}{Q(x)}$, we therefore need to integrate the simple elements of 1st and 2nd types.

Integration of ratios of 1st type $\frac{A}{(x-a)^l}, l\in\mathbb{N}^*$

- If $l=1$ then $\int\frac{A}{x-a}dx=A\ln|x-a|+c, c\in\mathbb{R}$.
- If $l>1$ then $\int\frac{A}{(x-a)^l}dx=\int A(x-a)^{-l}dx=\frac{A(x-a)^{-l+1}}{1-l}+c, c\in\mathbb{R}$.

Integration of ratios of 2nd type $\frac{Mx+N}{(x^2+px+q)^k}, k\ in\mathbb{N}^*$ with $p^2-4q<0$. We factorize the polynomial x^2+px+q in the form of the sum of two squares because $p^2-4q<0$, so we have

$$\begin{aligned}x^2+px+q&=\left(x+\frac{p}{2}\right)^2-\frac{p^2}{4}+q\\&=\left(x+\frac{p}{2}\right)^2+\frac{4q-p^2}{4}\end{aligned}$$

set $\frac{p}{2}=-\alpha$ and $\frac{4q-p^2}{4}=\beta^2$, therefore

$$x^2+px+q=(x-\alpha)^2+\beta^2$$

than replace

$$\begin{aligned}\frac{Mx+N}{\left(x^2+px+q\right)^k}&=\frac{Mx+N}{[(x-\alpha)^2+\beta^2]^k}\\&=\frac{Mx+N}{\beta^{2k}\left[\left(\frac{x-\alpha}{\beta}\right)^2+1\right]^k}\end{aligned}$$

the change of variable

$$t = \frac{x-\alpha}{\beta} \Leftrightarrow x = \beta t + \alpha \Rightarrow dx = \beta dt$$

transforms the above integral to

$$\begin{aligned}\int \frac{Mx+N}{\left(x^2+px+q\right)^k} dx &= \frac{1}{\beta^{2k-1}} \int \frac{M(\beta t+\alpha)+N}{\left[t^2+1\right]^k} dt \\ &= \frac{M}{\beta^{2k-2}} \int \frac{t}{\left[t^2+1\right]^k} dt + \frac{M\alpha+N}{\beta^{2k-1}} \int \frac{1}{\left[t^2+1\right]^k} dt\end{aligned}$$

therefore the integration of the rations of 2$^{\text{nd}}$ type: $\frac{Mx+N}{(x^2+px+q)^k}$ is reduced by the change of variables $x = \beta t + \alpha$ to the calculation of two primitives that we denote by:

$$I_k = \int \frac{t}{\left[t^2+1\right]^k} dt \text{ and } J_k = \int \frac{1}{\left[t^2+1\right]^k} dt$$

Calculation of I_k :

- If $k = 1$ then $I_1 = \int \frac{t}{t^2+1} dt = \frac{1}{2} \ln\left(t^2+1\right) + c, c \in \mathbb{R}$.
- If $k > 1$, so by the mean of the change of variable $u^2 = t^2 + 1 \Rightarrow 2udu = 2tdt$, we get

$$I_k = \int \frac{t}{\left[t^2+1\right]^k} dt = \int \frac{u}{u^{2k}} du = \int u^{1-2k} du = \frac{u^{2-2k}}{2-2k} + c, c \in \mathbb{R}.$$

thus

$$I_k = \frac{1}{2(1-k)u^{2k-2}} + c = \frac{1}{2(1-k)\left(t^2+1\right)^{k-1}} + c, c \in \mathbb{R}$$

Calculation of J_k :

- If $k = 1$ then $J_1 = \int \frac{1}{t^2+1} dt = \arctan t + c, c \in \mathbb{R}$
- If $k > 1$, so we integrate by parts, and we get

$$\begin{cases} u = \frac{1}{\left[t^2+1\right]^k} \\ dv = dt \end{cases} \Rightarrow \begin{cases} du = -2kt\left[t^2+1\right]^{-k-1} dt \\ v = t \end{cases}$$

so

$$J_k = \frac{t}{\left[t^2+1\right]^k} + 2k\int \frac{t^2+1-1}{\left[t^2+1\right]^{k+1}}dt$$
$$J_k = \frac{t}{\left[t^2+1\right]^k} + 2k\left[\int \frac{1}{\left[t^2+1\right]^k}dt - \int \frac{1}{\left[t^2+1\right]^{k+1}}dt\right]$$
$$J_k = \frac{t}{\left[t^2+1\right]^k} + 2kJ_k - 2kJ_{k+1}$$

then an induction formulas arise, which completes the calculations.

$$J_{k+1} = \frac{1}{2k}\left[\frac{t}{\left[t^2+1\right]^k} + (2k-1)J_k\right], k \in \mathbb{N}^*, k \geq 2.$$

Examples 8.4. *Integrate the following :*

1. $I_1 = \int \frac{x^2}{x^3-1}dx$

   Remark that

$$x^3 - 1 = (x-1)\left(x^2+x+1\right)$$

so

$$I_1 = \int \frac{x^2}{x^3-1}dx = \int \frac{x^2}{(x-1)\left(x^2+x+1\right)}dx$$

then

$$I_1 = \int \left[\frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}\right]dx$$

where A, B and C are real constants

$$\frac{x^2}{(x-1)\left(x^2+x+1\right)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}$$

elementary calculations give

$$\begin{aligned} I_1 &= \frac{1}{3}\int \frac{dx}{x-1} + \frac{1}{3}\int \frac{2x+1}{x^2+x+1}dx \\ &= \frac{1}{3}\left[\ln|x-1| + \ln\left(x^2+x+1\right)\right] + c, c \in \mathbb{R}, \\ &= \frac{1}{3}\ln\left|x^3-1\right| + c, c \in \mathbb{R}. \end{aligned}$$

2. $I_2 = \int \frac{x^3}{(x+1)\left(x^2+4\right)^2}dx$

   By factorization the rations, we get

$$I_2 = \int \left[\frac{A}{x+1} + \frac{Bx+C}{x^2+4} + \frac{Dx+E}{\left(x^2+4\right)^2}\right]dx$$

   where A, B, C, D and E are real constants, then we have

$$\frac{x^3}{(x+1)\left(x^2+4\right)^2}=\frac{A}{x+1}+\frac{Bx+C}{x^2+4}+\frac{Dx+E}{\left(x^2+4\right)^2}$$

for example , by identification we have

$$\begin{cases} A+B=0 \\ B+C=1 \\ 8A+4B+C+D=0 \\ 4B+4C+D+E=0 \\ 16A+4C+E=0 \end{cases}$$

resolving this system, gives

$$A=\frac{-1}{25}, B=\frac{1}{25}, C=\frac{24}{25}, D=\frac{-4}{5}, E=\frac{-16}{5}$$

so

$$I_2=\frac{-1}{25}\int\frac{dx}{x+1}+\frac{1}{25}\int\frac{x+24}{x^2+4}dx-\frac{4}{5}\int\frac{x+4}{\left(x^2+4\right)^2}dx$$

finally

$$\begin{aligned}\int\frac{x+24}{x^2+4}dx &= \frac{1}{2}\int\frac{2x}{x^2+4}dx+24\int\frac{1}{x^2+4}dx \\ &= \frac{1}{2}\ln\left(x^2+4\right)+12\int\frac{\frac{1}{2}}{\left(\frac{x}{2}\right)^2+1}dx \\ &= \frac{1}{2}\ln\left(x^2+4\right)+12\arctan\left(\frac{x}{2}\right)+c_1, c_1\in\mathbb{R}.\end{aligned}$$

and

$$\begin{aligned}\int\frac{x+4}{\left(x^2+4\right)^2}dx &= \frac{1}{2}\int\frac{2x}{\left(x^2+4\right)^2}dx+4\int\frac{1}{\left(x^2+4\right)^2}dx \\ &= \frac{1}{2}K+4L\end{aligned}$$

where

$$\begin{aligned}K=\int\frac{2x}{\left(x^2+4\right)^2}dx &= \int\frac{dt}{t^2}=-\frac{1}{t}+c_2, c_2\in\mathbb{R}. \\ &= -\frac{1}{x^2+4}+c_2,\quad c_2\in\mathbb{R}\end{aligned}$$

and

$$
\begin{aligned}
L &= \int \frac{1}{\left(x^2+4\right)^2} dx = \frac{1}{16} \int \frac{1}{\left(\left(\frac{x}{2}\right)^2+1\right)^2} dx \\
&= \frac{1}{8} \int \frac{1}{\left[t^2+1\right]^2} dt = \frac{1}{8} J_2, c_2 \in \mathbb{R},\ \textit{avec le CV} : t = \frac{x}{2}
\end{aligned}
$$

a suitable change of variable, gives

$$J_2 = \frac{1}{2}\left[\frac{t}{t^2+1} + J_1\right] = \frac{1}{2}\left[\frac{t}{t^2+1} + \arctan t\right] + c_3, c_3 \in \mathbb{R},$$

thus

$$L = \frac{1}{16}\left[\frac{t}{t^2+1} + \arctan t\right] + c_3, c_3 \in \mathbb{R}$$

and

$$L = \frac{1}{16}\left[\frac{2x}{x^2+4} + \arctan \frac{x}{2}\right] + c_3, c_3 \in \mathbb{R},$$

therefore,

$$\int \frac{x+4}{\left(x^2+4\right)^2} dx = -\frac{1}{2\left(x^2+4\right)} + \frac{1}{4}\left[\frac{2x}{x^2+4} + \arctan \frac{x}{2}\right] + c_3, c_3 \in \mathbb{R},$$

and, we conclude

$$I_2 = \frac{-1}{25} \ln|x+1| + \frac{1}{50} \ln\left(x^2+4\right) + \frac{7}{25} \arctan \frac{x}{2} + \frac{2(1-x)}{5\left(x^2+4\right)} + c, c \in \mathbb{R}.$$

8.2.4 Integration of irrational functions

Primitives of type $\int R\left(x, x^{\frac{k}{l}}, x^{\frac{m}{n}}, \ldots, x^{\frac{r}{s}}\right) dx$

To calculate this type of primitive, we first calculate α the least common multiple of denominators of the fractions $\frac{k}{l}, \frac{m}{n}, \ldots, \frac{r}{s}$, i.e.: $\alpha = \mathrm{LCM}(l, n, \ldots, s)$, then we make the change of variables

$$x = t^\alpha \Rightarrow dx = \alpha t^{\alpha-1} dt$$

Examples 8.5. *1. Express the primitive* : $I = \int \frac{\sqrt{x}}{x^{\frac{3}{4}}+1} dx = \int \frac{x^{\frac{1}{2}}}{x^{\frac{3}{4}}+1} dx$

We have $\alpha = LCM(2,4) = 4$, *the by the change of variable*

$$x = t^4 \Rightarrow dx = 4t^3 dt$$

so

$$I = 4\int \frac{t^5}{t^3+1}dt$$

and then

$$\begin{aligned}I &= 4\int \left[t^2 - \frac{t^2}{t^3+1}\right]dt = \frac{4}{3}\left[t^3 - \ln\left|t^3+1\right|\right] + c, c \in \mathbb{R},\\ &= \frac{4}{3}\left[x^{\frac{3}{4}} - \ln\left|x^{\frac{3}{4}}+1\right|\right] + c, c \in \mathbb{R}.\end{aligned}$$

Primitives of type $\int R\left(x, \left(\frac{ax+b}{cx+d}\right)^{\frac{k}{l}}, \left(\frac{ax+b}{cx+d}\right)^{\frac{m}{n}}, \ldots, \left(\frac{ax+b}{cx+d}\right)^{\frac{r}{s}}\right)dx$

To calculate this type of primitive, we first calculate α *least common multiple of denominators of the fractions* $\frac{k}{l}, \frac{m}{n}, \ldots, \frac{r}{s}$*, i.e.:*

$\alpha = \mathrm{LCM}(l, n, \ldots, s)$*, then we make the change of variables*

$$\frac{ax+b}{cx+d} = t^\alpha$$

Example 8.9. *The primitive of* $I = \int \frac{\sqrt{x+4}}{x}dx = \int \frac{(x+4)^{\frac{1}{2}}}{x}dx$

We have $\alpha = 2$*, the CV*

$$x + 4 = t^2 \Rightarrow dx = 2tdt$$

transforms to

$$\begin{aligned}I &= \int \frac{(x+4)^{\frac{1}{2}}}{x}dx = 2\int \frac{(t^2-4)+4}{t^2-4}dt = 2\left[\int dt + 4\int \frac{dt}{t^2-4}\right]\\ &= 2\left[t + \ln\left|\frac{t-2}{t+2}\right|\right] + c, c \in \mathbb{R},\\ &= 2\left[\sqrt{x+4} + \ln\left|\frac{\sqrt{x+4}-2}{\sqrt{x+4}+2}\right|\right] + c, c \in \mathbb{R}.\end{aligned}$$

Primitives of type $\int R\left(x, \sqrt{ax^2+bx+c}\right)dx, a \neq 0$

For this type of primitives we can use Euler Substitutions, where we distinguish two cases

1^{st} *Case If* $a > 0$ *then we make a change of variables by setting*

$$\sqrt{ax^2+bx+c} = \pm\sqrt{a}x + t$$

by making any choice of the sign before the root. Suppose that we choose the sign + for the rest of the calculations, then the square of the two sides of the last equation, we get

$$ax^2 + bx + c = ax^2 + t^2 + 2\sqrt{a}xt$$

which is equivalent to

$$x = \frac{t^2 - c}{b - 2\sqrt{a}t}.$$

Example 8.10. $I = \int \frac{x^2}{\sqrt{x^2+9}}dx$

One has $a = 1 > 0$, *the we set*

$$\sqrt{x^2+9} = x + t$$

so

$$x^2 + 9 = x^2 + 2xt + t^2 \Leftrightarrow x = \frac{9 - t^2}{2t}$$

therefore

$$\begin{aligned} dx &= \frac{(-2t)(2t) - (9 - t^2)\,2}{4t^2} \\ &= \frac{-2t^2 - 18}{4t^2}dt = \frac{-t^2 - 9}{2t^2}dt \end{aligned}$$

and

$$\sqrt{x^2+9} = \frac{9 + t^2}{2t}$$

we replace in I *and we obtain*

$$\begin{aligned} I &= -\frac{1}{4}\int \frac{\left(9 - t^2\right)^2}{t^3}dt = -\frac{1}{4}\int \frac{81 - 18t^2 + t^4}{t^3}dt \\ &= -\frac{1}{4}\left[81\int \frac{dt}{t^3} - 18\int \frac{dt}{t} + \int tdt\right] \\ &= -\frac{1}{4}\left[\frac{-81}{2t^2} - 18\ln|t| + \frac{t^2}{2}\right] + c, c \in \mathbb{R}. \end{aligned}$$

thus

$$I=-\frac{1}{4}\left[\frac{-81}{2\left(2x^2+9-2x\sqrt{x^2+9}\right)}-18\ln\left|\sqrt{x^2+9}-x\right|+x^2+\frac{9}{2}-x\sqrt{x^2+9}\right]+c, c\in\mathbb{R}.$$

2^{nd} *Case*

If $a<0$ and the discriminant Δ of the polynomial ax^2+bx+c is positive, i.e, $\Delta>0$ therefore the polynomial ax^2+bx+c admits two distinct real roots α and β, such that

$$ax^2+bx+c=a(x-\alpha)(x-\beta)$$

and in this case, we make another change of variables where we set

$$\sqrt{ax^2+bx+c}=(x-\alpha)t$$

here we can choose one of the two roots found either α or β, then we square the two sides of the equation

$$\sqrt{ax^2+bx+c}=(x-\alpha)t$$

while replacing the polynomial by its factorized form

$$ax^2+bx+c=a(x-\alpha)(x-\beta)$$

from where

$$a(x-\alpha)(x-\beta)=(x-\alpha)^2t^2$$

which is equivalent to

$$a(x-\beta)=(x-\alpha)t^2\Leftrightarrow x=\frac{a\beta-\alpha t^2}{a-t^2}.$$

Remark 8.3. *If the discriminant Δ of the polynomial ax^2+bx+c is null, and if the sign of a is positive, so the polynomial ax^2+bx+c admits a double real root α and we have*

$$ax^2+bx+c=a(x-\alpha)^2$$

thus

$$\sqrt{ax^2+bx+c}=\sqrt{a}|x-\alpha|.$$

Example 8.11. $I=\int\sqrt{2x-x^2}dx$
$\Delta>0\Rightarrow 2x-x^2=x(2-x)$, *we have* $\alpha=0$ *and* $\beta=2\overset{CV}{\Rightarrow}\sqrt{2x-x^2}=xt\Leftrightarrow x(2-x)=x^2t^2\Leftrightarrow x=\frac{2}{t^2+1}$ *then*

$$dx = \frac{-4t}{\left(t^2+1\right)^2} dt \ \ et \ \sqrt{2x-x^2} = \frac{2t}{t^2+1},$$

so

$$I = -8\int \frac{t^2}{\left(t^2+1\right)^3} dt = -8\int \frac{t^2+1-1}{\left(t^2+1\right)^3} dt$$
$$I = -8\int \frac{dt}{\left(t^2+1\right)^2} + 8\int \frac{dt}{\left(t^2+1\right)^3}$$
$$I = -8J_2 + 8J_3$$

where J_2 *et* J_3 *are defined before*

$$J_3 = \frac{1}{4}\left(\frac{t}{\left(t^2+1\right)^2} + 3J_2\right) = \frac{t}{4\left(t^2+1\right)^2} + \frac{3}{4}J_2$$

thus

$$I = -8J_2 + 8J_3 = \frac{2t}{\left(t^2+1\right)^2} - 2J_2 + c, c \in \mathbb{R}.$$

and

$$I = \frac{2t}{\left(t^2+1\right)^2} - 2J_2 + c, c \in \mathbb{R}.$$
$$I = \frac{2t}{\left(t^2+1\right)^2} - \frac{t}{t^2+1} - \arctan t + c, c \in \mathbb{R}.$$

where $t = \frac{\sqrt{2x-x^2}}{x}$, *therefore*

$$I = \frac{\sqrt{2x-x^2}}{2}(x-1) - \arctan\frac{\sqrt{2x-x^2}}{x} + c, c \in \mathbb{R}.$$

Remark 8.4. *If* $c > 0$, *then there exists an Euler substitution*

$$\sqrt{ax^2+bx+c} = xt \pm \sqrt{c}$$

there also the sign before the root remains at your choice, then we put the two sides of the equation

$$\sqrt{ax^2+bx+c} = xt \pm \sqrt{c}$$

squared, from where

$$ax^2 + bx + c = t^2x^2 + c + 2\sqrt{c}xt$$

which is equivalent to

$$x = \frac{t^2 - c}{b - 2\sqrt{a}t}.$$

Example 8.12. $I = \frac{dx}{x\sqrt{x^2-x+1}}$

$$\begin{aligned}
c = 1 > 0 &\overset{CV}{\Rightarrow} \sqrt{x^2 - x + 1} = xt + 1 \\
&\Rightarrow x^2 - x + 1 = (xt+1)^2 = x^2t^2 + 2xt + 1 \\
&\Rightarrow x = \frac{2t+1}{1-t^2} \Rightarrow dx = \frac{2\left(t^2+t+1\right)}{\left(1-t^2\right)^2} dt \\
&\textit{and } \sqrt{x^2 - x + 1} = \frac{t^2+t+1}{1-t^2}
\end{aligned}$$

so

$$\begin{aligned}
I &= \int \frac{dt}{2t+1} = \frac{1}{2}\ln|2t+1| + c, c \in \mathbb{R}. \\
I &= \frac{1}{2}\ln|2t+1| + c, c \in \mathbb{R}.
\end{aligned}$$

therefore

$$I = \frac{1}{2}\ln\left|\frac{2\sqrt{x^2-x+1} + x - 1}{x}\right| + c, c \in \mathbb{R}.$$

8.2.5 Integration of trigonometric functions

Primitives du type $\int R(\sin x, \cos x)dx$

There are several cases

Primitives of type $\int R(\cos x)\sin x dx$

In this case we put $t = \cos x$, so $dt = -\sin x dx$

Primitives of type $\int R(\sin x)\cos x dx$

We use the change of variable $t = \sin x$, thus $dt = \cos x dx$

Examples 8.6. *1.* $I = \int \frac{\cos^3 x}{\sin^4 x} dx$

$CV: t = \sin x \Rightarrow dt = \cos x dx$, *then*

$$\begin{aligned}
I &= \int \frac{1-t^2}{t^4} dt = \int \frac{dt}{t^4} - \int \frac{dt}{t^2} \\
&= -\frac{1}{3t^3} + \frac{1}{t} + c, c \in \mathbb{R}.
\end{aligned}$$

so

$$I = -\frac{1}{3\sin^3 x} + \frac{1}{\sin x} + c, c \in \mathbb{R}.$$

Primitives du type $\int R(\sin x, \cos x)dx$

The change of variable $t = \tan\frac{x}{2}$*, gives*

$$x = 2\arctan t \text{ then } dx = \frac{2}{1+t^2}dt$$

and

$$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} = 2\tan\frac{x}{2}\cos^2\frac{x}{2} = 2\frac{\tan\frac{x}{2}}{\tan^2\frac{x}{2}+1}$$

because

$$\sin^2\frac{x}{2} + \cos^2\frac{x}{2} = 1 \Leftrightarrow \tan^2\frac{x}{2} + 1 = \frac{1}{\cos^2\frac{x}{2}} \Leftrightarrow \cos^2\frac{x}{2} = \frac{1}{\tan^2\frac{x}{2}+1}$$

therefore

$$\sin x = \frac{2t}{t^2+1}$$

and

$$\cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2} = \cos^2\frac{x}{2}\left(1 - \tan^2\frac{x}{2}\right) = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$$

thus

$$\cos x = \frac{1-t^2}{1+t^2}.$$

Example 8.13. $I = \int \frac{dx}{\sin x}$

$CV: t = \tan\frac{x}{2} \Leftrightarrow x = 2\arctan t \Rightarrow dx = \frac{2}{1+t^2}dt$*, and we have* $\sin x = \frac{2t}{t^2+1}$

so

$$\begin{aligned} I &= 2\int \frac{dt}{t} = 2\ln|t| + c, c \in \mathbb{R}. \\ &= 2\ln\left|\tan\frac{x}{2}\right| + c, c \in \mathbb{R}. \end{aligned}$$

Primitives du type $\int R(\tan x)dx$

If the function to be integrated depends only on the tangent then we make the change of variables $t = \tan x$, hence $x = \arctan t$ and therefore $dx = \frac{1}{1+t^2}dt$.

Example 8.14. $I = \int tg^2 x dx$

$$CV : t = \tan x \Leftrightarrow x = \arctan t \Rightarrow dx = \frac{1}{1+t^2}dt$$

then

$$\begin{aligned} I &= \int \frac{t^2}{1+t^2}dt \\ &= \int \frac{(t^2+1)-1}{1+t^2}dt \\ &= \int dt - \int \frac{1}{1+t^2}dt \\ &= t - \arctan t + c, c \in \mathbb{R}. \end{aligned}$$

so

$$I = \tan x - x + c, c \in \mathbb{R}.$$

Primitives du type $\int R\left(\sin^n x, \cos^k x\right) dx$

n and k are two natural numbers. We suppose that they are even, in this case, we use

$$\begin{aligned} \sin^2 x &= \frac{1}{2}(1 - \cos 2x), \\ \cos^2 x &= \frac{1}{2}(1 + \cos 2x). \end{aligned}$$

Indeed, $\cos 2x = 1 - 2\sin^2 x = 2\cos^2 x - 1$. We can also use the following change of variable $t = \tan x$, then $x = \arctan t$ and therefore $dx = \frac{1}{1+t^2}dt$, one has

$$\cos^2 x = \frac{1}{1+\tan^2 x} = \frac{1}{1+t^2}$$

and

$$\sin^2 x = 1 - \cos^2 x = \frac{t^2}{1+t^2}$$

Examples 8.7. $I = \int \sin^4 x dx$

We have $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$, then

$$\begin{aligned} I &= \frac{1}{4}\int(1-\cos 2x)^2dx = \frac{1}{4}\int\left[1-2\cos 2x+(\cos 2x)^2\right]dx \\ &= \frac{1}{4}\left[x-\sin 2x+\int(\cos 2x)^2dx\right] \end{aligned}$$

but, $(\cos 2x)^2 = \frac{1}{2}(1+\cos 4x)$, *so*

$$\int(\cos 2x)^2dx = \frac{1}{2}\int(1+\cos 4x)dx = \frac{1}{2}\left(x+\frac{1}{4}\sin 4x\right)+c, c\in\mathbb{R}.$$

thus

$$I = \frac{1}{4}\left[x-\sin 2x+\frac{1}{2}\left(x+\frac{1}{4}\sin 4x\right)\right]+c, c\in\mathbb{R}.$$

therefore,

$$I = \frac{3}{4}x-\frac{1}{4}\sin 2x+\frac{1}{32}\sin 4x+c, c\in\mathbb{R}.$$

$I = \int\frac{dx}{\cos^4 x}$

$$CV: t=\tan x \Leftrightarrow x=\arctan t \Rightarrow dx=\frac{1}{1+t^2}dt \;\; et \;\; on \;\; a\cos^2 x=\frac{1}{1+t^2}, d' \;\; où$$

$$\begin{aligned} I &= \int\left(1+t^2\right)dt = t+\frac{t^3}{3}+c \\ &= \tan x+\frac{(\tan x)^3}{3}+c, c\in\mathbb{R}. \end{aligned}$$

Primitives $\int\cos kx\cos nx dx, \int\sin kx\cos nx dx, \int\sin kx\sin nx dx$

For this kind of primitives, we use the well known trigonometric formulas

$$\begin{aligned} \cos(a+b) &= \cos a\cos b-\sin a\sin b \\ \cos(a-b) &= \cos a\cos b+\sin a\sin b \\ \sin(a-b) &= \cos a\sin b-\sin a\cos b \end{aligned}$$

to have the following transformations

$$\begin{aligned} \cos kx\cdot\cos nx &= \frac{1}{2}[\cos(k+n)x+\cos(k-n)x] \\ \sin kx\cdot\cos nx &= \frac{1}{2}[\cos(k+n)x+\sin(k-n)x] \\ \sin kx\cdot\sin nx &= \frac{1}{2}[-\cos(k+n)x+\cos(k-n)x] \end{aligned}$$

Example 8.15. $I = \int \sin 5x \cdot \sin 3x dx$.

We apply the formula

$$\sin 5x \cdot \sin 3x = \frac{1}{2}[-\cos 8x + \cos 2x]$$

from where

$$\begin{aligned} I &= \frac{1}{2}\int [-\cos 8x + \cos 2x] dx \\ &= \frac{1}{4}\left[-\frac{1}{4}\sin 8x + \sin 2x\right] + c, c \in \mathbb{R}. \end{aligned}$$

8.2.6 Integration of certain irrational functions by the mean of trigonometric transformations

We consider primitives of the type $\int R\left(x, \sqrt{ax^2+bx+c}\right) dx$, *with* $a \neq 0, ax^2+bx+c > 0$, *and* $\Delta \neq 0$.

We start by rewriting it under canonical decomposition,

$$ax^2 + bx + c = a\left[\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}\right],$$

then we make the change of variables $t = x + \frac{b}{2a}$, *so* $dt = dx$ *and from there we come back to one of the following three forms:*

- $\int R\left(t, \sqrt{n^2t^2 + k^2}\right) dt$ *where we make the change of variables* $t = \frac{k}{n}\tan z$.
- $\int R\left(t, \sqrt{n^2t^2 - k^2}\right) dt$, *such that* $n^2t^2 - k^2 \geq 0$, *where we make the change of variables* $t = \frac{k}{n \sin z}$.
- $\int R\left(t, \sqrt{k^2 - n^2t^2}\right) dt$, *such as* $k^2 - n^2t^2 \geq 0$, *where we make the change of variables* $t = \frac{k}{n}\sin z$.

Example 8.16. $I = \int \frac{dx}{\sqrt{(-x^2-2x)^3}}$.
We have

$$-x^2 - 2x = -\left[(x+1)^2 - 1\right] = 1 - (x+1)^2$$

the change of variables $t = x + 1$, *gives* $dt = dx$ *from where we get*

$$I = \int \frac{dt}{\sqrt{(1-t^2)^3}}$$

A second change of variables $t = \sin z \Rightarrow dt = \cos z dz$, *so*

$$I = \int \frac{dz}{\cos^2 z} = \tan z + c, c \in \mathbb{R}.$$

but

$$\tan z = \frac{\sin z}{\cos z} = \frac{t}{\sqrt{1-t^2}}$$

thus

$$I = \frac{t}{\sqrt{1-t^2}} + c$$

therefore

$$I = \frac{x+1}{\sqrt{-x^2-2x}} + c, c \in \mathbb{R}.$$

CHAPTER

9

DEFINITE INTEGRALS

9.1 The Riemann Integral

Now, we give the definition of the Riemann integral of a function.

Definition 9.1. *Let $[a, b]$ be a closed interval.*

(1) The set $P = \{x_0, x_1, x_2, \ldots, x_n\}$ of finite points $a = x_0, x_1, x_2, \ldots, x_n = b$ with

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b$$

is called a partition of $[a, b]$;

(2) For every $i = 1, 2, \ldots, n$, a closed interval $I_i = [x_{i-1}, x_i]$ is called the subinterval of $[a, b]$;

(3) For every $i = 1, 2, \ldots, n$, the length of $I_i = [x_{i-1}, x_i]$ is defined by $\Delta x_i = x_i - x_{i-1}$,

(4) $\wp[a, b]$ is denoted by the set of all partitions of $[a, b]$,

(5) The width of the largest sub-interval in a partition is called the norm of the partition.

9.1.1 Step functions

Definition 9.2. *A function $f : [a, b] \to \mathbb{R}$is called a step function if it is piecewise constant, i.e. if there are numbers $a = x_0 < x_1 < x_2 < ... < x_{N-1} < x_N = b$ such that* *f is constant on each half open interval*$[x_{i-1}, x_i]$ *with* $1 \leqslant i \leqslant N$. *For a step function we define the integral to be* $\int_a^b f(x)dx = \sum_{i=1}^N f(x_{i-1})(x_i - x_{i-1})$. *The collection of numbers* $x_0, x_1, x_2, \dots x_N$ *are called a partition for the step function* f .

In this definition we have to take into account that a step function f might be defined using different partitions, and show that the integral does not depend on which partition is used to compute it.

Lemma 9.1. *If* f *and* g *are step functions on an interval* $[a, b]$ *with* $f(x) \leqslant g(x)$ *for all* $x \in [a, b]$, *then* $\int_a^b f(x)dx \leq \int_a^b g(x)dx$.

Definition 9.3. *Let* $f : [a, b] \to \mathbb{R}$ *be a bounded function and* $P = \{a = x_0, x_1, x_2, \dots, x_n = b\}$ *be a partition of* $[a, b]$. *For every subinterval* $I_i = [x_{i-1}, x_i]$. *We define Riemann sum* $S(f, P) = \sum_{i=1}^n f(x_i^\star)\Delta x_i$, *where* $\Delta x_i = x_i - x_{i-1}$ *and* $x_i^* \in [x_{i-1}, x_i]$.

Remark 9.1. *One might produce different Riemann sums depending on which* x_i^**'s are chosen. In the end this will not matter, if the function is Riemann integrable, when the difference or width of the summands* Δx_i *approaches zero.*

Types of Riemann sums

Specific choices of x_i^* *give different types of Riemann sums:*

1. *If* $x_i^* = x_{i-1}$ *for all i, the method is the left rule and gives a left Riemann sum.*
2. *If* $x_i^* = x_i$ *for all i, the method is the right rule and gives Riemann sum.*
3. *If* $x_i^* = (x_i + x_{i-1})/2$ *for all* i, *the method is the midpoint rule and gives a middle Riemann sum.*
4. *If* $f(x_i^*) = \sup f([x_{i-1}, x_i])$ *(that is, the supremum of* f *over* $[x_{i-1}, x_i]$*), the method is the upper rule and gives an upper Riemann sum or upper Darboux sum.*
5. *If* $f(x_i^*) = \inf f([x_{i-1}, x_i])$ *(that is, the infimum of* f *over* $[x_{i-1}, x_i]$*), the method is the lower rule and gives a lower Riemann sum or lower Darboux sum.*

Definition 9.4. *Let* $f : [a, b] \to \mathbb{R}$ *be a bounded function and* $P = \{a = x_0, x_1, x_2, \dots, x_n = b\}$ *be a partition of* $[a, b]$. *For every subinterval* $I_i = [x_{i-1}, x_i]$, *put*

$$M_i = \sup \{f(x) : x \in [x_{i-1}, x_i]\}, \quad m_i = \inf \{f(x) : x \in [x_{i-1}, x_i]\}$$

Then, for every $P \in \wp[a, b]$, *define the upper Riemann sum* $U(f, P)$ *and the lower Riemann sum* $L(f, P)$ *as follows, respectively:*

$$U(f, P) = \sum_{i=1}^n M_i \Delta x_i, \quad L(f, P) = \sum_{i=1}^n m_i \Delta x_i$$

Example 9.1. *Let $f : [a, b] \to \mathbb{R}$ be a function defined by $f(x) = c$, where $c \in \mathbb{R}$ is a constant. Prove that, for every $P = \{a = x_0, x_1, x_2, \ldots, x_n = b\} \in \wp[a, b]$*

$$U(f, P) = L(f, P) = c(b - a)$$

Solution. Let $\Delta x_i = x_i - x_{i-1}$ for every $i = 1, 2, \ldots, n$. Since $M_i = m_i = c$, we have the following:

$$U(f, P) = \sum_{i=1}^{n} M_i \Delta x_i = \sum_{i=1}^{n} c\Delta x_i = c\sum_{i=1}^{n} \Delta x_i = c(b - a)$$

and

$$L(f, P) = \sum_{i=1}^{n} m_i \Delta x_i = \sum_{i=1}^{n} c\Delta x_i = c\sum_{i=1}^{n} \Delta x_i = c(b - a).$$

Thus, we have

$$U(f, P) = L(f, P) = c(b - a)$$

Example 9.2. *Let $f : [0, 1] \to \mathbb{R}$ be a function defined by*

$$f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q} \\ 0, & \text{if } x \notin \mathbb{Q}^c \end{cases}$$

Then, calculate $U(f, P)$ and $L(f, P)$.

Solution. Let $P = \{0 = x_0, x_1, x_2, \ldots, x_n = 1\} \in \wp[0, 1]$. Each subinterval $I_i = [x_{i-1}, x_i]$ for every $i = 1, 2, \ldots, n$ has infinitely many rational numbers and irrational numbers, and so, for every $i = 1, 2, \ldots, n$,

$$M_i = \sup\{f(x) : x \in [x_{i-1}, x_i]\} = 1$$

and

$$m_i = \inf\{f(x) : x \in [x_{i-1}, x_i]\} = 0.$$

Hence, we have $L(f, P) = 0$ and $U(f, P) = 1$.

Lemma 9.2. *Let $f : [a, b] \to \mathbb{R}$ be a bounded function and $P = \{a = x_0, x_1, x_2, \ldots, x_n = b\}$ be a partition of $[a, b]$. Then, we have the following:*

(1) $L(f, P) \leq U(f, P)$;

(2) $m(b-a) \leq L(f,P) \leq U(f,P) \leq M(b-a)$, where $m, M \in \mathbb{R}$ are numbers such that

$$m \leq f(x) \leq M \text{ for every } x \in [a,b]$$

Definition 9.5. *Let $[a,b]$ be a closed interval.*

(1) For every $P, Q \in \wp[a,b], Q$ is called the refinement of P if

$$P \subset Q$$

(2) The partition $R = P \cup Q$ is called the common refinement of P and Q.

Lemma 9.3. *Let $f : [a,b] \to \mathbb{R}$ be a bounded function. For every $P, Q \in \wp[a,b]$, if Q is a refinement of P, then we have the following:*

(1) $L(f,P) \leq L(f,Q)$

(2) $U(f,Q) \leq U(f,P)$.

Proof. (1) First, let

$$P = \{a = x_0, x_1, x_2, \ldots, x_n = b\}$$

and

$$Q = \{a = x_0, x_1, x_2, \ldots, x_{j-1}, x^*, x_j, \ldots, x_n = b\}$$

that is, $Q = P \cup \{x^*\}$. Then, Q is a refinement of P, that is, $P \subset Q$. Next, put

$$m_j' = \inf\{f(x) : x \in [x_{j-1}, x^*]\}, m_j'' = \inf\{f(x) : x \in [x^*, x_j]\}$$

and

$$m_j = \min\left\{m_j', m_j''\right\}$$

Then, we have

$$L(f,P) = \sum_{i=1}^{n} m_i \Delta x_i = \sum_{i=1, i\neq j}^{n} m_i \Delta x_i + m_j \Delta x_j$$

and

$$L(f,Q) = \sum_{i=1}^{j-1} m_i \Delta x_i + m_j' (x^* - x_{j-1}) + m_j'' (x_j - x^*) + \sum_{i=j+1}^{n} m_i \Delta x_i.$$

Thus, since $m_j \leq m_j', m_j''$, we have

$$
\begin{aligned}
m_j \Delta x_j &= m_j\left(x^* - x_{j-1}\right) + m_j\left(x_j - x^*\right) \\
&\leq m_j'\left(x^* - x_{j-1}\right) + m_j''\left(x_j - x^*\right)
\end{aligned}
$$

and so

$$
\begin{aligned}
\sum_{i=1, i\neq j}^{n} m_j \Delta x_j &= \sum_{i=1, i\neq j}^{n} m_j\left(x^* - x_{j-1}\right) + \sum_{i=1, i\neq j}^{n} m_j\left(x_j - x^*\right) \\
&\leq \sum_{i=1, i\neq j}^{n} m_j'\left(x^* - x_{j-1}\right) + \sum_{i=1, i\neq j}^{n} m_j''\left(x_j - x^*\right),
\end{aligned}
$$

which implies that $L(f, P) \leq L(f, Q)$.

Next, let Q be the refinement adjoining a finite points to P, and then, repeating the proceeding process, we can prove $L(f, P) \leq L(f, Q)$.

(2) By the same argument as in (1), we can prove $U(f, Q) \leq U(f, P)$. This completes the proof. □

Theorem 9.4. *Let $f : [a, b] \to \mathbb{R}$ be a bounded function. For every $P, Q \in \wp[a, b]$, (Q isn't a refinement of P), we have $L(f, P) \leq U(f, Q)$.*

Proof. Let $T = P \cup Q$. Then, T is a refinement of P and Q. Thus, by Lemma (9.3), we have

$$
L(f, P) \leq L(f, T), \quad U(f, T) \leq U(f, Q).
$$

Therefore, since $L(f, P) \leq U(f, P)$, we have

$$
L(f, P) \leq U(f, Q).
$$

This completes the proof. □

Corollary 9.1. *Let $f : [a, b] \to \mathbb{R}$ be a bounded function. For every $P, Q \in \wp[a, b]$, if $L(f, P) \leq U(f, Q)$, then*

$$
\sup\{L(f, P) : P \in \wp[a, b]\}, \quad \inf\{U(f, P) : P \in \wp[a, b]\}
$$

exist.

Proof. Let $Q \in \wp[a, b]$ be fixed. Then, for every $P \in \wp[a, b]$, we have

$$
L(f, P) \leq U(f, Q), \quad L(f, Q) \leq U(f, P),
$$

which means that $U(f, Q)$ and $L(f, Q)$ are upper bound and lower bound of the following sets:

$$\{L(f,P) : P \in \wp[a,b]\}, \quad \{U(f,P) : P \in \wp[a,b]\},$$

respectively. Therefore, by the Completeness Axiom of $\mathbb{R}$, it follows that

$$\sup\{L(f,P) : P \in \wp[a,b]\}, \quad \inf\{U(f,P) : P \in \wp[a,b]\}$$

exist. This completes the proof. $\square$

Corollary 9.2. *Let $f, g : [a,b] \to \mathbb{R}$ be bounded functions and $c \in \mathbb{R}$. For every $P \in \wp[a,b]$, we have the following:*

(1) $U(f+g,P) \leq U(f,P) + U(g,P)$

(2) $L(f+g,P) \geq L(f,P) + L(g,P)$;

(3) If $c \geq 0$, then we have

$$U(cf,P) = cU(f,P), \quad L(cf,P) = cL(f,P)$$

(4) If $c < 0$, then we have

$$U(cf,P) = cL(f,P), \quad L(cf,P) = cU(f,P).$$

Proof. Let $P = \{a = x_0, x_1, x_2, \ldots, x_n = b\} \in \wp[a,b]$.

(1) We have

$$\begin{aligned} U(f+g,P) &= \sum_{i=1}^{n} \sup\{f(x)+g(x) : x \in [x_{i-1},x_i]\}\,\Delta x_i \\ &\leq \sum_{i=1}^{n} [\sup\{f(x) : x \in [x_{i-1},x_i]\} \\ &\qquad + \sup\{g(x) : x \in [x_{i-1},x_i]\}]\,\Delta x_i \\ &= U(f,P) + U(g,P) \end{aligned}$$

(2) As in the proof of (1), we have $L(f+g,P) \geq L(f,P) + L(g,P)$.

(3) Let $c \geq 0$. Then, we have

$$\begin{aligned} U(cf,P) &= \sum_{i=1}^{n} \sup\{cf(x) : x \in [x_{i-1},x_i]\}\,\Delta x_i \\ &= c\sum_{i=1}^{n} \sup\{f(x) : x \in [x_{i-1},x_i]\}\,\Delta x_i \\ &= cU(f,P) \end{aligned}$$

(4) Let $c < 0$. Then, we have

$$
\begin{aligned}
U(cf,P) &= \sum_{i=1}^{n} \sup\{cf(x) : x \in [x_{i-1}, x_i]\}\,\Delta x_i \\
&= c\sum_{i=1}^{n} \inf\{f(x) : x \in [x_{i-1}, x_i]\}\,\Delta x_i \\
&= cL(f,P).
\end{aligned}
$$

This completes the proof. □

Definition 9.6. *Let $f : [a,b] \to \mathbb{R}$ be a bounded function.*

(1) The upper Riemann integral of f on $[a,b]$ is the number

$$\overline{\int_a^b} f(x)dx = \inf\{U(f,P) : P \in \wp[a,b]\}$$

(2) The lower Riemann integral of f on $[a,b]$ is the number

$$\underline{\int_a^b} f(x)dx = \sup\{L(f,P) : P \in \wp[a,b]\}$$

(3) f is said to be Riemann integrable on $[a,b]$ if

$$\overline{\int_a^b} f(x)dx = \underline{\int_a^b} f(x)dx$$

(4) In this case, the Riemann integral of f is defined to be the number

$$\overline{\int_a^b} f(x)dx = \underline{\int_a^b} f(x)dx$$

which is denoted by

$$\int_a^b f(x)dx$$

(5) In addition, we define

$$\int_a^b f(x)dx = -\int_b^a f(x)dx, \quad \int_a^a f(x)dx = 0$$

(6) f is said to be not Riemann integrable on $[a,b]$ if

$$\overline{\int_a^b} f(x)dx \neq \underline{\int_a^b f(x)dx}$$

Remark 9.2. *Note that $\Re[a,b]$ denotes the set of Riemann integrable functions on $[a,b]$.*

From the definition of the Riemann integral, it follows that, for every $p, Q \in \wp[a,b]$ with $P \subset Q$

$$L(f,P) \leq L(f,Q) \leq \int_a^b f(x)dx \leq U(f,Q) \leq U(f,P)$$

Example 9.3. *Let* $f : [a,b] \to \mathbb{R}$ *be the function defined by*

$$f(x) = \begin{cases} 1, & \textit{if } x \in \mathbb{Q} \\ 0, & \textit{if } x \notin \mathbb{Q}^c \end{cases}$$

Then, we have $L(f,P) = 0$ *and* $U(f,P) = 1$ *and so*

$$\overline{\int_a^b} f(x)dx = \inf\{U(f,P) : P \in \wp[a,b]\} = 1$$

and

$$\underline{\int_a^b} f(x)dx = \sup\{L(f,P) : P \in \wp[a,b]\} = 0.$$

Therefore, sine $\overline{\int_a^b} f(x)dx = 1 \neq 0 = \underline{\int_a^b} f(x)dx, f$ *is not Riemann integrable on* $[0,1]$.

Theorem 9.5. *(The Riemann Integral Test) Let* $f : [a,b] \to \mathbb{R}$ *be a bounded function. Then, the following are equivalent:*

(1) $f \in \Re[a,b]$

(2) For every $\varepsilon > 0$*, there exists* $P \in \wp[a,b]$ *such that*

$$U(f,P) - L(f,P) < \varepsilon$$

Proof. (1) $\Longrightarrow$ (2) Suppose that $f \in \Re[a,b]$. Since

$$\int_a^b f(x)dx = \inf\{U(f,P) : P \in \wp[a,b]\}$$

for every $\varepsilon > 0, \int_a^b f(x)dx + \frac{\varepsilon}{2}$ is not a lower bound of $\{U(f,P) : P \in \wp[a,b]\}$, and so there exists $P_1 \in \wp[a,b]$ such that

$$U(f,P_1) < \int_a^b f(x)dx + \frac{\varepsilon}{2}$$

Also, since

$$\int_a^b f(x)dx = \sup\{L(f,P) : P \in \wp[a,b]\}$$

$\int_a^b f(x)dx - \frac{\varepsilon}{2}$ is not an upper bound of $\{L(f,P) : P \in \wp[a,b]\}$, and so there exists $P_2 \in \wp[a,b]$ such that

$$\int_a^b f(x)dx - \frac{\varepsilon}{2} < L(f,P_2)$$

Letting $P = P_1 \cup P_2$, we have

$$L(f, P_2) \leq L(f, P) \leq U(f, P) \leq U(f, P_1)$$

Thus, we have

$$\begin{aligned} U(f, P) - L(f, P) &\leq U(f, P_1) - L(f, P_2) \\ &< \left(\int_a^b f(x)dx + \frac{\varepsilon}{2}\right) - \left(\int_a^b f(x)dx - \frac{\varepsilon}{2}\right) \\ &= \varepsilon. \end{aligned}$$

(2) $\Longrightarrow$ (1) Suppose that, for every $\varepsilon > 0$, there exists $P \in \wp[a, b]$ such that

$$U(f, P) - L(f, P) < \varepsilon$$

Since $\overline{\int_a^b} f(x)dx \leq U(f, P)$ and $L(f, P) \leq \underline{\int_a^b} f(x)dx$, we have

$$0 \leq \overline{\int_a^b} f(x)dx - \underline{\int_a^b} f(x)dx \leq U(f, P) - L(f, P) < \varepsilon$$

and so

$$|\overline{\int_a^b} f(x)dx - \underline{\int_a^b} f(x)dx f(x)dx| = \overline{\int_a^b} f(x)dx - \underline{\int_a^b} f(x)dx < \varepsilon.$$

Therefore, we have

$$\overline{\int_a^b} f(x)dx = \underline{\int_a^b} f(x)dx$$

and $f \in \Re[a, b]$. This completes the proof

□

Example 9.4. *Let $f : [a, b] \to \mathbb{R}$ be the function defined by $f(x) = x$ for every $x \in [a, b]$. By using the definition of the Riemann integral, show that*

$$\int_a^b f(x)dx = \frac{1}{2}\left(b^2 - a^2\right)$$

Solution. For every $n \geq 1$, let P_n be a partition of $[a, b]$ given by

$$P_n = \left\{a, a + \frac{b-a}{n}, a + \frac{2(b-a)}{n}, \ldots, a + \frac{n(b-a)}{n} = b\right\}.$$

Since $f(x) = x$ is increasing on $[a, b]$, at each subinterval $\left[a + \frac{(i-1)(b-a)}{n}, a + \frac{i(b-a)}{n}\right]$, f has the minimum value $m_i = a + \frac{(i-1)(b-a)}{n}$ and the maximum value $M_i = a + \frac{i(b-a)}{n}$. Therefore, we have

$$\begin{aligned}
L(f, P_n) &= \sum_{i=1}^{n} m_i \Delta x_i \\
&= \sum_{i=1}^{n} \left(a + \frac{(i-1)(b-a)}{n}\right)\left(\frac{b-a}{n}\right) \\
&= a(b-a) + \frac{(b-a)^2}{2}\left(1 - \frac{1}{n}\right).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
U(f, P_n) &= \sum_{i=1}^{n} M_i \Delta x_i \\
&= \sum_{i=1}^{n} \left(a + \frac{i(b-a)}{n}\right)\left(\frac{b-a}{n}\right) \\
&= a(b-a) + \frac{(b-a)^2}{2}\left(1 + \frac{1}{n}\right)
\end{aligned}$$

which implies that $f(x) = x$ is Riemann integrable on $[a, b]$ and

$$\int_a^b f(x)dx = \frac{1}{2}\left(b^2 - a^2\right)$$

Corollary 9.3. *Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Then, the following are equivalent:*

(1) $f \in \Re[a, b]$;

(2) There exists a sequence $\{P_n\}$ of partitions of $[a, b]$

$$\lim_{n\to\infty} \left(U(f, P_n) - L(f, P_n)\right) = 0.$$

In this case, we have

$$\lim_{n\to\infty} L(f, P_n) = \int_a^b f(x)dx = \lim_{n\to\infty} U(f, P_n).$$

Example 9.5. *Let $f : [0, 1] \to \mathbb{R}$ be the function defined by $f(x) = x$ for every $x \in [0, 1]$. By using the Riemann Integral Test, prove that $f(x) = x$ is Riemann integrable on $[0, 1]$.*

Solution. Let $P_n \in \wp[0, 1]$ as in Example 3.4 be a partition of $[a, b]$. Then, we have

$$L(f, P_n) = \frac{n-1}{2n}, \quad U(f, P_n) = \frac{n+1}{2n}.$$

For every $\varepsilon > 0$, if we take a positive integer n_0 such that

$$\frac{1}{\varepsilon} < n_0$$

then we have

$$U\left(f, P_{n_0}\right)-L\left(f, P_{n_0}\right)=\frac{1}{n_0}<\varepsilon .$$

Therefore, by Theorem (the Riemann Integral Test), f is Riemann integrable on $[0,1]$.

A function $f:[a, b] \rightarrow \mathbb{R}$ is said to be monotone on $[a, b]$ if it is either increasing or decreasing on $[a, b]$.

Theorem 9.6. *Every monotone function $f:[a, b] \rightarrow \mathbb{R}$ is Riemann integrable.*

Proof. Assume that f is increasing on $[a, b]$. Since $f(a) \leq f(x) \leq f(b)$ for every $x \in [a, b]$, f is clearly bounded on $[a, b]$. Let $\varepsilon>0$, and select a positive integer n such that

$$\frac{(f(b)-f(a))(b-a)}{n}<\varepsilon .$$

For a partition

$$P=\left\{a=x_0<x_1<\cdots<x_{n-1}<x_n=b\right\},$$

where $\Delta x_i=x_i-x_{i-1}=\frac{b-a}{n}$ for every $i \geq 1$, we have

$$U(f, P)=\sum_{i=1}^n M_i \Delta x_i=\sum_{i=1}^n f\left(x_i\right) \frac{b-a}{n}$$

and

$$L(f, P)=\sum_{i=1}^n m_i \Delta x_i=\sum_{i=1}^n f\left(x_{i-1}\right) \frac{b-a}{n} .$$

Thus, since f is increasing on $[a, b]$, we have

$$\begin{aligned} U(f, P)-L(f, P) & =\frac{b-a}{n} \sum_{i=1}^n\left|f\left(x_i\right)-f\left(x_{i-1}\right)\right| \\ & =\frac{b-a}{n}(f(b)-f(a)) \\ & <\varepsilon . \end{aligned}$$

Therefore, by the Riemann Integral Test, f is Riemann integrable on $[a, b]$.

Similarly, we can prove this theorem when f is decreasing on $[a, b]$. This completes the proof.

□

Theorem 9.7. *Let $f:[a, b] \rightarrow \mathbb{R}$ be a continuous function. Then, $f \in \Re[a, b]$, that is, f is Riemann integrable on $[a, b]$.*

Proof. Since f is a continuous function on $[a,b]$, f is uniformly continuous on $[a,b]$. Thus, for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|x-y| < \delta, x, y \in [a,b] \Longrightarrow |f(x) - f(y)| < \frac{\varepsilon}{b-a}$$

Now, select a sufficiently large positive integer n, and consider a partition $P = \{x_0, x_1, x_2, \ldots, x_n\}$ of $[a,b]$ with

$$\Delta x_1 = \Delta x_2 = \cdots = \Delta x_{n_0} = \frac{b-a}{n}$$

Further, for each subinterval $I_i = [x_{i-1}, x_i]$ of the partition P_n $(1 \le i \le n)$, there exist $t_i, u_i \in I_i$ such that

$$M_i = \sup\{f(x) : x \in [x_{i-1}, x_i]\} = f(t_i)$$

and

$$m_i = \inf\{f(x) : x \in [x_{i-1}, x_i]\} = f(u_i)$$

Thus, we have

$$\begin{aligned}
0 &\le U(f, P_n) - L(f, P_n) \\
&= \sum_{i=1}^{n} (M_i - m_i)\, \Delta x_i \\
&= \sum_{i=1}^{n} (f(t_i) - f(u_i))\, \Delta x_i \\
&= \sum_{i=1}^{n_0} |f(t_i) - f(u_i)| \left(\frac{b-a}{n}\right) \\
&< \frac{\varepsilon}{b-a}(b-a) \\
&= \varepsilon
\end{aligned}$$

Therefore, by the Riemann Integral Test, f is Riemann integrable on $[a,b]$. This completes the proof.

□

9.2 Properties of the Riemann Integral

In this section, we give the Riemann integrals of the composition and the product of two Riemann integrable functions and some basic properties of the Riemann integral including some algebraic properties of the Riemann integral.

Theorem 9.8. *Let $f : [a,b] \to \mathbb{R}$ be Riemann integrable on $[a,b]$ and $g : [c,d] \to \mathbb{R}$ be a continuous function on $[c,d]$ with $f([a,b]) \subset [c,d]$. Then, the composition $g \circ f$ is Riemann integrable on $[a,b]$.*

Proof. Let $\varepsilon > 0$. Since g is continuous on $[c,d]$, let

$$M = \sup\{f(x) : x \in [c,d]\}$$

and let

$$\varepsilon' = \frac{\varepsilon}{2M + (b-a)}.$$

Also, g is uniformly continuous on $[c,d]$, and there exists $\delta > 0$ such that $0 < \delta < \varepsilon'$ and

$$y, z \in [c,d], |y - z| \leq \delta \Longrightarrow |g(y) - g(z)| < \varepsilon'.$$

On the other hand, since f is Riemann integrable on $[a,b]$, there exists a partition $P = \{a = x_0, x_1, x_2, \ldots, x_n = b\}$ of $[a,b]$ such that

$$U(f,P) - L(f,P) < \delta^2.$$

Now, we show that, for this partition P,

$$U(g \circ f, P) - L(g \circ f, P) \leq \varepsilon$$

and then $g \circ f$ is Riemann integrable on $[a,b]$. For every $i = 1, 2, \ldots, n$, put

$$\begin{aligned}
M_i &= \sup\left\{f(x) : x \in [x_{i-1}, x_i]\right\}, \\
m_i &= \inf\left\{f(x) : x \in [x_{i-1}, x_i]\right\}, \\
M_i' &= \sup\left\{(g \circ f)(x) : x \in [x_{i-1}, x_i]\right\}, \\
m_i' &= \inf\left\{(g \circ f)(x) : x \in [x_{i-1}, x_i]\right\}.
\end{aligned}$$

Now, to be convenient, we separate the indices of the partition P into two disjoint subsets X and Y as follows:

$$X = \left\{k : M_i - m_i < \delta\right\}, \quad Y = \left\{i : M_i - m_i \geq \delta\right\}.$$

Now, if $i \in X$, that is, $M_i - m_i < \delta$, then, for every $x, x' \in [x_{i-1}, x_i]$, we have

$$|f(x) - f(x')| < \delta$$

and so

$$|(g \circ f)(x) - (g \circ f)(x')| < \varepsilon'$$

which implies that

$$M_i' - m_i' \leq \varepsilon$$

Thus, we have

$$\sum_{i \in X} (M_i' - m_i')(x_i - x_{i-1}) \leq \varepsilon'(b-a). \tag{9.1}$$

But, if $i \in Y$, then we have $\delta \leq M_i - m_i$ and so

$$\begin{aligned} \sum_{i \in Y} (x_i - x_{i-1}) &\leq \frac{1}{\delta} \sum_{i \in Y} (M_i - m_i)(x_i - x_{i-1}) \\ &\leq \frac{1}{\delta}(U(f,P) - L(f,P)) \\ &< \delta < \varepsilon' \end{aligned}$$

Therefore, we have

$$\sum_{i \in Y} (M_i' - m_i')(x_i - x_{i-1}) \leq 2M\varepsilon'. \tag{9.2}$$

If we combine (9.1) and (9.2) then we have

$$\begin{aligned} &U(g \circ f, P) - L(g \circ f, P) \\ &= \sum_{i \in X} (M_i' - m_i')(x_i - x_{i-1}) + \sum_{i \in Y} (M_i' - m_i')(x_i - x_{i-1}) \\ &\leq \varepsilon'(b-a) + 2M\varepsilon' \\ &= \varepsilon. \end{aligned}$$

Therefore, by the Riemann Integral Test, $g \circ f$ is Riemann integrable on $[a,b]$. This completes the proof.

□

Theorem 9.9. *Let $f, g : [a,b] \to \mathbb{R}$ be Riemann integrable on $[a,b]$, that is, $f, g \in \Re[a,b]$, and let $c \in \mathbb{R}$. Then, we have the following:*

(1) $f + g \in \Re[a,b]$ and

$$\int_a^b (f+g)(x)dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

(2) $cf \in \Re[a,b]$ *and*

$$\int_a^b cf(x)dx = c\int_a^b f(x)dx$$

(3) $fg \in \Re[a,b]$

(4) If $f(x) \leq g(x)$ *for every* $x \in [a,b]$*, then*

$$\int_a^b f(x)dx \leq \int_a^b g(x)dx$$

(5) If $|f| \in \Re[a,b]$ *then* $\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx$

Proof. (1) Since $f, g \in \Re[a,b]$, for every $\varepsilon > 0$, there exist $P_1, P_2 \in \wp[a,b]$ such that

$$U(f,P_1) - L(f,P_1) < \frac{\varepsilon}{2}, \quad U(g,P_2) - L(g,P_2) < \frac{\varepsilon}{2}$$

Put $P = P_1 \cup P_2$. Then, we have $P \in \wp[a,b]$ and

$$\begin{aligned} U(f+g,P) - L(f+g,P) &\leq U(f,P) + U(g,P) - L(f,P) - L(g,P) \\ &= U(f,P) - L(f,P) + U(g,P) - L(g,P) \\ &\leq U(f,P_1) - L(f,P_1) + U(g,P_2) - L(g,P_2) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

Therefore, by the Riemann Integral Test, we have $f + g \in \Re[a,b]$.

On the other hand, it follows that, for every $P \in \wp[a,b]$,

$$\begin{aligned} L(f,P) + L(g,P) &\leq L(f+g,P) \\ &\leq \int_a^b (f+g)(x)dx \\ &\leq U(f+g,P) \\ &\leq U(f,P) + U(g,P). \end{aligned} \tag{9.3}$$

Observe that

$$U(f,P) \leq U(f,P_1) < L(f,P_1) + \frac{\varepsilon}{2} \leq \int_a^b f(x)dx + \frac{\varepsilon}{2}$$

and

$$U(g,P) \leq U(g,P_2) < L(g,P_2) + \frac{\varepsilon}{2} \leq \int_a^b g(x)dx + \frac{\varepsilon}{2}$$

Thus, by (9.3), we have

$$
\begin{aligned}
\int_a^b (f+g)(x)dx &\leq U(f+g,P) \\
&\leq U(f,P)+U(g,P) \\
&< \int_a^b f(x)dx + \int_a^b g(x)dx + \varepsilon
\end{aligned}
$$

Since $\varepsilon > 0$ is arbitrary, we have

$$
\int_a^b (f+g)(x)dx \leq \int_a^b f(x)dx + \int_a^b g(x)dx \tag{9.4}
$$

Similarly, we have

$$
\begin{aligned}
\int_a^b (f+g)(x)dx &\geq L(f+g,P) \\
&\geq L(f,P)+L(g,P) \\
&\geq U(f,P)-\frac{\varepsilon}{2}+U(g,P)-\frac{\varepsilon}{2} \\
&> \int_a^b f(x)dx + \int_a^b g(x)dx - \varepsilon
\end{aligned}
$$

Which leads to

$$
\int_a^b (f+g)(x)dx \geq \int_a^b f(x)dx + \int_a^b g(x)dx \tag{9.5}
$$

Therefore, from (9.4) and (9.5), it follows that

$$
\int_a^b (f+g)(x)dx = \int_a^b f(x)dx + \int_a^b g(x)dx
$$

(2) It is easy to prove (2) by the definition of the Riemann integral.

(3) Let $f \in \Re[a,b]$ and $h(x) = x^2$ for all $x \in [a,b]$. Then, we have

$$
h \circ f = f^2 \in \Re[a,b].
$$

Next, let $f, g \in \Re[a,b]$. Then, by (1) and (2), we have

$$
-g, f+g, f-g, (f+g)^2, (f-g)^2 \in \Re[a,b].
$$

Therefore, since

$$
fg = \frac{1}{4}\left[(f+g)^2-(f-g)^2\right]
$$

we have $fg \in \Re[a,b]$.

(4) Just consider $H(x) = g(x) - f(x) \geq 0$

(5) Since $g(x) = |x|$ is continuous on $[a,b]$, we have $|f| = g \circ f \in \Re[a,b]$.

On the other hand, let $c \in \mathbb{R}$ ($c = \pm 1$) be such that

$$c\int_a^b f(x)dx = \left|\int_a^b f(x)dx\right|$$

Then, since $cf(x) \leq |f(x)|$ for every $x \in [a,b]$, we have

$$\int_a^b cf(x)dx \leq \int_a^b |f(x)|dx$$

Therefore, by (2), we have

$$\left|\int_a^b f(x)dx\right| = c\int_a^b f(x)dx = \int_a^b cf(x)dx \leq \int_a^b |f(x)|dx.$$

This completes the proof.

□

From the theorem above, we know that

$$\Re[a,b] = \{f : [a,b] \to \mathbb{R} : \text{ a Riemann integrable function on } [a,b]\}$$

is a vector space.

Theorem 9.10. *Let $f : [a,b] \to \mathbb{R}$ be a bounded function on $[a,b]$, and let a number c with $a < c < b$. Then, the following are equivalent:*

(1) f is Riemann integrable on $[a,b]$;

(2) f is Riemann integrable on both $[a,c]$ and $[c,b]$. In this case, we have

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

Proof. (2) $\Longrightarrow$ (1) Suppose that f is Riemann integrable on both $[a,c]$ and $[c,b]$. Then, by the Riemann Integral Test, for every $\varepsilon > 0$, there exist $P_1 \in \wp[a,c]$ and $P_2 \in \wp[c,b]$ such that

$$U(f,P_1) - L(f,P_1) < \frac{\varepsilon}{2}$$

and

$$U(f,P_2) - L(f,P_2) < \frac{\varepsilon}{2}$$

Let $P = P_1 \cup P_2$. Then, we have

$$U(f,P) - L(f,P) = [U(f,P_1) + U(f,P_2)] - [L(f,P_1) + L(f,P_2)]$$
$$= [U(f,P_1) - L(f,P_1)] + [U(f,P_2) - L(f,P_2)] < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, by the Riemann Integral Test, f is Riemann integrable on $[a, b]$.

(1) $\Longrightarrow$ (2) Suppose that f is Riemann integrable on $[a, b]$. Then, by the Riemann Integral Test, for every $\varepsilon > 0$, there exists $P_1 \in \wp[a, b]$ such that

$$U(f, P_1) - L(f, P_1) < \varepsilon$$

If $P = P_1 \cup \{c\}$, then P is a refinement of P_1 and so

$$U(f, P) - L(f, P) \leq U(f, P_1) - L(f, P_1) < \varepsilon \tag{9.6}$$

Let $P_2 = P \cap [a, c]$ and $P_3 = P \cap [c, b]$. Then, for $P = P_2 \cup P_3$, we have

$$U(f, P) = U(f, P_2) + U(f, P_3), \quad L(f, P) = L(f, P_2) + L(f, P_3)$$

and so, from (9.6), it follows that

$$[U(f, P_2) - L(f, P_2)] + [U(f, P_3) - L(f, P_3)] < \varepsilon$$

Thus, we have

$$U(f, P_2) - L(f, P_2) < \varepsilon, U(f, P_3) - L(f, P_3) < \varepsilon \tag{9.7}$$

Since $\varepsilon > 0$ is arbitrary, by the Riemann Integral Test, f is Riemann integrable on both $[a, c]$ and $[c, b]$.

Finally, we prove

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

In fact, from (9.7), it follows that

$$\begin{aligned}\int_a^b f(x)dx &\leq U(f, P) = U(f, P_2) + U(f, P_3) \\ &< L(f, P_2) + L(f, P_3) + 2\varepsilon \\ &\leq \int_a^c f(x)dx + \int_c^b f(x)dx + 2\varepsilon\end{aligned} \tag{9.8}$$

Also, from from (9.7), it follows that

$$\begin{aligned}\int_a^c f(x)dx+\int_c^b f(x)dx &\leq U(f,P_2)+U(f,P_3)\\ &< L(f,P_2)+L(f,P_3)+2\varepsilon\\ &= L(f,P)+2\varepsilon\\ &\leq \int_a^b f(x)dx+2\varepsilon\end{aligned}\tag{9.9}$$

Since $\varepsilon>0$ is arbitrary, it follows from (9.8) and (9.9) that

$$\int_a^b f(x)dx=\int_a^c f(x)dx+\int_c^b f(x)dx.$$

This completes the proof.

□

9.3 The Fundamental Theorems of Calculus

In this section, we prove the Fundamental Theorem of Calculus for a correction between the derivative and the Riemann integral.

Theorem 9.11. *Let $f:[a,b]\to\mathbb{R}$ be Riemann integrable on $[a,b]$, and define a function $F:[a,b]\to\mathbb{R}$ by*

$$F(x)=\int_a^x f(t)dt \quad \textit{for every } x\in[a,b]$$

Then, we have the following:

(1) F is uniformly continuous on $[a,b]$;

(2) If f is continuous at $x_0\in[a,b]$, then F is differentiable at x_0 and $F'(x_0)=f(x_0)$

Proof. (1) Since f is bounded on $[a,b]$, there exists a constant $M>0$ such that

$$|f(t)|\leq M \text{ for every } t\in[a,b]$$

Further, for every $x,y\in[a,b]$, we have

$$|F(x)-F(y)|=\left|\int_x^y f(t)dt\right|\leq\int_x^y|f(t)|dt\leq M|y-x|.$$

Therefore, for every $\varepsilon>0$, if $\delta=\frac{\varepsilon}{M}$, then

$$|y-x|<\delta, x,y\in[a,b]\Longrightarrow|F(y)-F(x)|<\varepsilon$$

which implies that F is uniformly continuous on $[a, b]$.
(2) Let $\varepsilon > 0$. Since f is continuous at $x = x_0$, there exists $\delta > 0$ such that

$$|x - x_0| < \delta, x \in [a, b] \Longrightarrow |f(x) - f(x_0)| < \varepsilon$$

Thus, if $0 < |x - x_0| < \delta$, then we have

$$\begin{aligned} \left|\frac{F(x) - F(x_0)}{x - x_0} - f(x_0)\right| &= \left|\frac{1}{x - x_0}\left(\int_0^x f(t)dt - \int_0^{x_0} f(t)dt\right) - f(x_0)\right| \\ &\leq \frac{1}{x - x_0}\int_{x_0}^x |f(t) - f(x_0)|\, dt < \varepsilon \end{aligned}$$

Therefore, F is differentiable at x_0, and $F'(x_0) = f(x_0)$. This completes the proof.

□

Theorem 9.12. *(The Fundamental Theorem of Calculus I) Let $f : [a, b] \to \mathbb{R}$ be a continuous function on $[a, b]$, and define a function $F : [a, b] \to \mathbb{R}$ by*

$$F(x) = \int_a^x f(t)dt \quad \text{for every } x \in [a, b].$$

Then, F is differentiable on $[a, b]$, and $F'(x) = f(x)$.

Proof. Since f is continuous on $[a, b]$, for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|t - x| < \delta \Longrightarrow |f(t) - f(x)| < \varepsilon.$$

Let h be a number with $0 < h < \delta$. Then, we have

$$F(x+h) - F(x) = \int_a^{x+h} f(t)dt - \int_a^x f(t)dt = \int_x^{x+h} f(t)dt$$

and so

$$\begin{aligned} \left|\frac{F(x+h) - F(x)}{h} - f(x)\right| &= \left|\frac{1}{h}\int_x^{x+h} f(t)dt - \frac{1}{h}\int_x^{x+h} f(x)dt\right| \\ &\leq \frac{1}{h}\int_x^{x+h} |f(t) - f(x)|dt < \varepsilon. \end{aligned}$$

Therefore, we have

$$F'_+(x) = \lim_{h \to 0^+} \frac{F(x+h) - F(x)}{h} = f(x).$$

Similarly, for $-\delta < h < 0$, we have

$$F'_-(x) = \lim_{h \to 0^-} \frac{F(x+h) - F(x)}{h} = f(x).$$

Since $F_{+}'(x) = F_{-}'(x) = f(x), F$ is differentiable on $[a,b]$ and, for all $x \in [a,b]$, $F'(x) = f(x)$. This completes the proof.

□

Theorem 9.13. *(The Fundamental Theorem of Calculus II) Let $f : [a,b] \to \mathbb{R}$ be Riemann integrable on $[a,b]$ and F be differentiable on $[a,b]$. If $F'(x) = f(x)$, then we have*

$$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$$

Proof. Let $P = \{a = x_0, x_1, x_2, \ldots, x_n = b\}$ be a partition of $[a,b]$. Then, if we apply Mean Value Theorem to F at each subinterval $I_i = [x_{i-1}, x_i]$ for every $i = 1, 2, \ldots, n$, then it follows that there exists $t_i \in [x_{i-1}, x_i]$ such that

$$F(x_i) - F(x_{i-1}) = f(t_i)(x_i - x_{i-1})$$

Thus, we have

$$F(b) - F(a) = \sum_{i=1}^{n} (F(x_i) - F(x_{i-1})) = \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}).$$

Put

$$M_i = \sup\{f(x) : x \in [x_{i-1}, x_i]\}, \quad m_i = \inf\{f(x) : x \in [x_{i-1}, x_i]\}.$$

Then, we have $m_i \le f(t_i) \le M_i$ for every $i = 1, 2, \ldots, n$ and so

$$\sum_{i=1}^{n} m_i(x_i - x_{i-1}) \le F(b) - F(a) \le \sum_{i=1}^{n} M_i(x_i - x_{i-1}).$$

Thus, it follows that, for every $P \in \wp[a,b]$,

$$L(f,P) \le F(b) - F(a) \le U(f,P)$$

and so, since

$$\sup\{L(f,P) : P \in \wp[a,b]\} \le F(b) - F(a) \le \inf\{U(f,P) : P \in \wp[a,b]\}$$

and $f \in \Re[a,b]$, we have

$$\int_a^b f(x)dx = F(b) - F(a).$$

This completes the proof.

□

Example 9.6. *Find an example in which if F is not differentiable on $[a,b]$, then Theorem (9.13) is not true.*

Solution. *Define two functions $f, F : [0,1] \to \mathbb{R}$ by $f(x) = 1$ for every $x \in [0,1]$ and*

$$F(x) = \begin{cases} x, & \text{if } x \in [0,1), \\ 0, & \text{if } x = 1, \end{cases}$$

respectively. Then, f is Riemann integrable on $[0,1]$ and $\int_0^1 f(x)dx = 1$.

On the other hand, we have $F(1) = F(0) = 0$, and, for every $x \in [0,1)$, $F'(x) = f(x)$, but F is not differentiable at $x = 1$. Thus, the Theorem (9.13) is not true.

Theorem 9.14. *(The Generalized Mean Value Theorem for Integral.) Let $f : [a,b] \to \mathbb{R}$ be a continuous function on $[a,b]$ and $g : [a,b] \to \mathbb{R}$ be Riemann integrable on $[a,b]$ with $g(x) \geq 0$ for every $x \in [a,b]$. Then, there exists $c \in [a,b]$ such that*

$$\int_a^b f(x)g(x)dx = f(c)\int_a^b g(x)dx.$$

Proof. Since $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$, let

$$M = \sup\{f(x) : x \in [a,b]\}, \quad m = \inf\{f(x) : x \in [a,b]\}$$

Then, it follows that

$$mg(x) \leq f(x)g(x) \leq Mg(x) \text{ for every } x \in [a,b]$$

and fg is Riemann integrable on $[a,b]$ and so

$$m\int_a^b g(x)dx \leq \int_a^b f(x)g(x)dx \leq M\int_a^b g(x)dx \tag{9.10}$$

If $\int_a^b g(x)dx = 0$, then the conclusion follows easily.

If $\int_a^b g(x)dx \neq 0$, then it follows from (9.10) that

$$m \leq \frac{\int_a^b f(x)g(x)dx}{\int_a^b g(x)dx} \leq M$$

and so, since f is continuous on $[a,b]$, by the Intermediate Value Theorem, there exists $c \in [a,b]$ such that

$$f(c) = \frac{\int_a^b f(x)g(x)dx}{\int_a^b g(x)dx}$$

that is,

$$\int_a^b f(x)g(x)dx = f(c)\int_a^b g(x)dx$$

This completes the proof.

$\square$

Example 9.7. *Define two functions $f, g : [-1,1] \to \mathbb{R}$ by*

$$f(x) = x, \quad g(x) = e^x \text{ for every } x \in [-1,1]$$

Then, apply Theorem (9.14) to the functions f and g on $[-1,1]$.

Solution. Since f and g are continuous on $[-1,1]$, they are Riemann integrable on $[-1,1]$. Further, $g(x) > 0$ for every $x \in [-1,1]$ and

$$\int_{-1}^1 f(x)g(x)dx = \frac{2}{e}, \quad \int_{-1}^1 g(x)dx = \frac{e^2-1}{e}$$

and so

$$\frac{2}{e^2-1} = \frac{\int_{-1}^1 f(x)g(x)dx}{\int_{-1}^1 g(x)dx}$$

Since f is continuous on $[-1,1]$ and

$$f(-1) = -1 < \frac{2}{e^2-1} < 1 = f(1)$$

by the Intermediate Value Theorem, there exists $c \in [-1,1]$ such that

$$c = f(c) = \frac{2}{e^2-1}$$

Therefore, we have

$$f(c)\int_{-1}^1 g(x)dx = \frac{2}{e^2-1} \cdot \frac{e^2-1}{e} = \frac{2}{e} = \int_{-1}^1 f(x)g(x)dx.$$

Corollary 9.4. *(The Mean Value Theorem for Integral) Let $f : [a,b] \to \mathbb{R}$ be a continuous function on $[a,b]$. Then, there exists $c \in [a,b]$ such that*

$$\int_a^b f(x)dx = f(c)(b-a)$$

Proof. Proof. In Theorem (9.14), if $g(x) = 1$ for every $x \in [a,b]$, then we have

$$\int_a^b f(x)dx = f(c)(b-a)$$

$\square$

Example 9.8. *By using the Mean Value Theorem for Integral, prove the following inequalities:*

$$\frac{x}{1+x} < \ln(1+x) < x \text{ for every } x > 0$$

Solution. *Let* $f(x) = \frac{1}{1+x}$ *for every* $x > 0$*. Then, we have*

$$\int_0^x \frac{1}{1+t}dt = \ln(1+x)$$

Thus, by Corollary (9.4), there exists $c \in [0, x]$ *such that*

$$\int_0^x \frac{1}{1+t}dt = f(c)(x-0)$$

Since f *is decreasing on* $[0, x]$ *and*

$$\inf\{f(t) : t \in [0, x]\} \leq f(c) \leq \sup\{f(t) : t \in [0, x]\}$$

we have

$$\frac{1}{1+x} < \frac{\ln(1+x)}{x} < 1 \text{ for every } x > 0$$

which implies that

$$\frac{x}{1+x} < \ln(1+x) < x \text{ for every } x > 0$$

9.3.1 The Substitution Theorem and Integration by Parts

In this section, we prove the Substitution Theorem and the Integration by Parts as techniques of integration, which are based on the Fundamental Theorems of Calculus.

Theorem 9.15. *(The Substitution Theorem) Let* $f : [a, b] \to \mathbb{R}$ *be a continuous function on* $[a, b]$ *and* $g : [c, d] \to [a, b]$ *be differentiable on* $[c, d]$*. If* $g(c) = a$ *and* $g(d) = b$*, then we have*

$$\int_a^b f(x)dx = \int_c^d f(g(t))g'(t)dt$$

Proof. For every $x \in [a, b]$, define a function $F : [a, b] \to \mathbb{R}$ by

$$F(x) = \int_a^x f(t)dt$$

Then, by Theorem (The Fundamental Theorem of Calculus I), F is differentiable on $[a, b]$. Thus, we have

$$(F(g(t)))' = F'(g(t))g'(t) = f(g(t))g'(t).$$

Therefore, by Theorem (The Fundamental Theorem of Calculus II), we have

$$\int_c^d f(g(t))g'(t)dt = F(g(d)) - F(g(c)) = F(b) - F(a) = \int_a^b f(x)dx$$

This completes the proof.

□

Example 9.9. *Let* $f : [0,1] \to \mathbb{R}$ *be a continuous function on* $[0,1]$. *Evaluate the following:*

$$\int_{-1}^{1} xf\left(x^2\right) dx$$

Solution. Let $\phi(x) = x^2$ *for every* $x \in [-1,1]$. *Then, since we have* $\phi([-1,1]) = [0,1]$ *and* f *is continuous on* $[0,1]$, *we have*

$$\int_{-1}^{1} f\left(x^2\right) dx = \frac{1}{2}\int_{-1}^{1} f(\phi(x))\phi'(x)dx = \frac{1}{2}\int_{1}^{1} f(t)dt = 0$$

Theorem 9.16. *(The Integration by Parts) Let* $f, g : [a,b] \to \mathbb{R}$ *be differentiable on* $[a,b]$ *and* f', g' *be Riemann integrable on* $[a,b]$. *Then, we have*

$$\int_a^b f(x)g'(x)dx = f(b)g(b) - f(a)g(a) - \int_a^b f'(x)g(x)dx$$

Proof. Since f and g are differentiable on $[a,b]$, fg is also differentiable on $[a,b]$ and so, for every $x \in [a,b]$,

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x).$$

Since f and g are continuous on $[a,b]$, they are Riemann integrable on $[a,b]$. Since f' and g' are Riemann integrable on $[a,b]$, $(fg)' = f'g + fg'$ is also Riemann integrable on $[a,b]$. Therefore, by Theorem (The Fundamental Theorem of Calculus I), we have

$$\int_a^b (f(x)g(x))'dx = f(b)g(b) - f(a)g(a)$$

and so

$$\int_a^b f'(x)g(x)dx + \int_a^b f(x)g'(x)dx = f(b)g(b) - f(a)g(a),$$

which implies that

$$\int_a^b f(x)g'(x)dx = f(b)g(b) - f(a)g(a) - \int_a^b f'(x)g(x)dx.$$

This completes the proof.

□

Example 9.10. *Evaluate the following:*

$$\int_0^{\frac{\pi}{2}} \cos^3 x dx$$

Solution. Now, we have

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \cos^3 x dx &= \int_0^{\frac{\pi}{2}} \cos^2 x(\cos x dx) \\ &= \int_0^{\frac{\pi}{2}} \left(1 - \sin^2 x\right)(\cos x dx).\end{aligned}$$

Let $u = \sin x$*. Then, we have* $du = \cos x dx$*. If* $x = 0$*, then* $u = 0$*. If* $x = \frac{\pi}{2}$*, then* $u = 1$*. Therefore, we have*

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \cos^3 x dx &= \int_0^1 \left(1 - u^2\right) du \\ &= \left[u - \frac{1}{3}u^3\right]_0^1 \\ &= 1 - \frac{1}{3} = \frac{2}{3}.\end{aligned}$$

Theorem 9.17 (Luxemburg Monotone Convergence Theorem)**.** *Let* $\{f_n(x)\}$ *be a decreasing sequence of bounded functions on* $[a, b]$*, with* $a < b$*, which converges pointwise to 0 on* $[a, b]$*. Then*

$$\lim_{n\to\infty} \underline{\int_a^b} f_n(x)dx = 0.$$

Proof. The lower integral of any function is well defined provided the function is bounded. Therefore for any bounded function $f(x)$, and by definition of the lower integral, for any $\varepsilon > 0$, there exists a step function $L(x) \leq f(x)$ such that

$$\underline{\int_a^b}(f(x) - L(x))dx < \frac{\varepsilon}{2}$$

Also note the existence of a continuous function $c(x) \leq L(x)$ such that

$$\int_a^b (L(x) - c(x))dx < \frac{\varepsilon}{2}$$

Putting all this together, we conclude that for any bounded function $f(x)$, and for any $\varepsilon > 0$, there exists a continuous function $c(x) \leq f(x)$ such that

$$\underline{\int_a^b}(f(x) - c(x))dx < \varepsilon$$

Note that if $f(x) \geq 0$, then the construction of $c(x)$ will be done to have $c(x) \geq 0$ as well. Back to our claim. Let $\varepsilon > 0$. Then there exists a positive continuous function $c_1(x) \leq f_1(x)$ such that

$$\underline{\int_a^b} (f_1(x) - c_1(x))\, dx < \frac{\varepsilon}{2^2}$$

Since $\min(c_1(x), f_2(x))$ is bounded and positive, there exists a positive continuous function $c_2(x) \leq \min(c_1(x), f_2(x))$ such that

$$\underline{\int_a^b} (\min(c_1(x), f_2(x)) - c_2(x))\, dx < \frac{\varepsilon}{2^3}$$

Since $f_2(x)-c_2(x) \leq f_2(x)-\min(c_1(x), f_2(x))+\min(c_1(x), f_2(x))-c_2(x)$, and $f_2(x) \leq f_1(x)$, we get

$$\begin{aligned} \underline{\int_a^b} (f_2(x) - c_2(x))\, dx &\leq \underline{\int_a^b} (f_1(x) - c_1(x))\, dx \\ &\quad + \underline{\int_a^b} (\min(c_1(x), f_2(x)) - c_2(x))\, dx < \frac{\varepsilon}{2^2} + \frac{\varepsilon}{2^3}. \end{aligned}$$

By the induction argument, a similar construction will lead to the existence of a decreasing sequence of positive continuous functions $\{c_n(x)\}$ such that $c_n(x) \leq f_n(x)$ and

$$\underline{\int_a^b} (f_n(x) - c_n(x))\, dx < \frac{\varepsilon}{2^2} + \frac{\varepsilon}{2^3} + \cdots + \frac{\varepsilon}{2^{n+1}} < \frac{\varepsilon}{2}.$$

Since $\{f_n(x)\}$ converges pointwise to 0 on $[a, b]$ this will force the sequence $\{c_n(x)\}$ to also converge pointwise to 0 on $[a, b]$. Dini's theorem will imply that $\{c_n(x)\}$ converges uniformly to 0 on $[a, b]$. So there exists $n_0 \geq 1$ such that for any $n \geq n_0$ we have $c_n(x) \leq \frac{\varepsilon}{2(b-a)}$ for any $x \in [a, b]$. Hence

$$\underline{\int_a^b} f_n(x)dx \leq \underline{\int_a^b} (f_n(x) - c_n(x))\, dx + \int_a^b c_n(x)dx < \frac{\varepsilon}{2} + \frac{\varepsilon}{2(b-a)}(b-a) = \varepsilon$$

whenever $n \geq n_0$. This finishes the proof of our claim. □

Theorem 9.18 (Monotone Convergence Theorem)**.** *Let $\{f_n(x)\}$ be a decreasing sequence of Riemann integrable functions on $[a, b]$ which converges pointwise to a Riemann integrable function $f(x)$. Then*

$$\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx$$

Proof. Since $f(x)$ is Riemann integrable, $\{f_n(x) - f(x)\}$ is a sequence of Riemann integrable functions which decreases to 0 . Obviously they are all bounded functions. The Luxemburg Monotone Convergence Theorem will imply

$$\lim_{n\to\infty} \underline{\int_a^b} (f_n(x) - f(x))\, dx = 0.$$

But $\underline{\int_a^b} (f_n(x) - f(x))\, dx = \int_a^b (f_n(x) - f(x))\, dx$, hence

$$\lim_{n\to\infty} \int_a^b (f_n(x) - f(x))\, dx = 0$$

which implies

$$\lim_{n\to\infty} \int_a^b f_n(x) = \int_a^b f(x)dx$$

□

Theorem 9.19 (Arzelà Theorem)**.** *Let $\{f_n(x)\}$ be a sequence of Riemann integrable functions on $[a,b]$ which converges pointwise to a Riemann integrable function $f(x)$. Assume that there exists $M > 0$ such that $|f_n(x)| \leq M$ for any $x \in [a,b]$ and $n \geq 1$. Then*

$$\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx$$

Proof. Since $f(x)$ is Riemann integrable, $\{f_n(x) - f(x)\}$ is a sequence of Riemann integrable functions which decreases to 0 . Obviously they are all bounded functions. The Luxemburg Monotone Convergence Theorem will imply

$$\lim_{n\to\infty} \underline{\int_a^b} (f_n(x) - f(x))\, dx = 0.$$

But $\underline{\int_a^b} (f_n(x) - f(x))\, dx = \int_a^b (f_n(x) - f(x))\, dx$, hence

$$\lim_{n\to\infty} \int_a^b (f_n(x) - f(x))\, dx = 0$$

which implies

$$\lim_{n\to\infty} \int_a^b f_n(x) = \int_a^b f(x)dx$$

□

Lemma 9.20 (Fatoo)**.** *Let $\{f_n(x)\}$ be a sequence of Riemann integrable functions on $[a,b]$ which converges pointwise to a Riemann integrable function $f(x)$. Then*

$$\int_a^b f(x)dx \leq \liminf_{n\to\infty} \int_a^b f_n(x)dx$$

Proof. If $\liminf_{n\to\infty} \int_a^b f_n(x)dx = \infty$, then the conclusion is obvious.
Assume that $\liminf_{n\to\infty} \int_a^b f_n(x)dx < \infty$. Then there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that

$$\lim_{n_k\to\infty} \int_a^b f_{n_k}(x)dx = \liminf_{n\to\infty} \int_a^b f_n(x)dx$$

Clearly the subsequence $\{f_{n_k}\}$ also converges pointwise to $f(x)$. Set $h_{n_i}(x) = \inf_{n_k \geq n_i} f_{n_k}(x)$. Then $\{h_{n_k}(x)\}$ also converges pointwise to $f(x)$ and is increasing. It is easy to see that this sequence is bounded. The Luxemburg Monotone Convergence Theorem applied to $\{f(x) - h_{n_k}(x)\}$ will easily imply that

$$\lim_{n\to\infty} \int_a^b h_{n_k}(x)dx = \int_a^b f(x)dx$$

Since $h_{n_k}(x) \leq f_{n_k}(x)$, we get

$$\int_a^b f(x)dx \leq \lim_{n\to\infty} \int_a^b f_{n_k}(x)dx = \liminf_{n\to\infty} \int_a^b f_n(x)dx$$

□

9.3.2 Hermite Hadamard Inequality

Let $f : I \longrightarrow \mathbb{R}$ be a convex function defined on the interval I , for the two real numbers a, b of I with $a < b$, the following double inequality :

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)dx \leqslant \frac{f(a)+f(b)}{2} \tag{9.11}$$

Well known in the literature as the Hermite-Hadamard inequality, it gives us an estimate of the mean value of the convex function.

If the function f is concave, the inequality becomes

$$f\left(\frac{a+b}{2}\right) \geqslant \frac{1}{b-a}\int_a^b f(x)dx \geqslant \frac{f(a)+f(b)}{2} \tag{9.12}$$

9.4 The Hermite-Hadamard Inequality Refined and Simply Proved

Lemma 9.21. *Let f be an integrable function over I. We have:*

$$\frac{1}{b-a}\int_a^b f(x)dx = \int_0^1 f(\lambda b+(1-\lambda)a)d\lambda \tag{9.13}$$

$$= \int_0^1 f(\lambda a+(1-\lambda)b)d\lambda \tag{9.14}$$

Proof. We use the change of variable $x=\lambda b+(1-\lambda)a$ to prove (9.13) and $x=\lambda a+(1-\lambda)b$ to prove (9.14).

for $x=a$, $a=\lambda b+(1-\lambda)a \implies \lambda=0$

for $x=b$, $b=\lambda b+(1-\lambda)a \implies \lambda=1$

We integrate f with respect to λ over $[0,1]$ we obtain (9.13).

for $x=b$, $b=\lambda a+(1-\lambda)b \implies \lambda=0$

for $x=a$, $a=\lambda a+(1-\lambda)b \implies \lambda=1$

We integrate f with respect to λ over $[0,1]$ we obtain (9.14). □

9.4.1 Proof of the inequality of Hermite Hadamard

Thanks to the convexity of f we have for all $\lambda\in[0,1]$:

$$\begin{aligned}
f\left(\frac{a+b}{2}\right) &= f\left(\frac{\lambda b+(1-\lambda)a+\lambda a+(1-\lambda)b}{2}\right)\\
&\leqslant \frac{f(\lambda b+(1-\lambda)a)+f(\lambda a+(1-\lambda)b)}{2}\\
&\leqslant \frac{1}{2}\left[f(\lambda b+(1-\lambda)a)+f(\lambda a+(1-\lambda)b)\right]\\
&\leqslant \frac{1}{2}\left[\lambda f(b)+f(a)-\lambda f(a)+\lambda f(a)+f(b)-\lambda f(b)\right]\\
&\leqslant \frac{1}{2}\left[f(a)+f(b)\right]\\
&\leqslant \frac{f(a)+f(b)}{2}
\end{aligned}$$

So we can write:

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{f(\lambda b+(1-\lambda)a)+f(\lambda a+(1-\lambda)b)}{2} \leqslant \frac{f(a)+f(b)}{2} \tag{9.15}$$

We integrate the inequality (9.15) over $[0,1]$:

$$\int_0^1 f\left(\frac{a+b}{2}\right)d\lambda \leqslant \int_0^1 \frac{f\left(\lambda b+(1-\lambda)a+f(\lambda a+(1-\lambda)b\right)}{2}d\lambda \leqslant \int_0^1 \frac{f(a)+f(b)}{2}d\lambda$$

We get:

$$f\left(\frac{a+b}{2}\right) \leqslant \int_0^1 \frac{f\left(\lambda b+(1-\lambda)a\right)+f(\lambda a+(1-\lambda)b)}{2}d\lambda \leqslant \frac{f(a)+f(b)}{2}$$

Using Lemma 9.21:

$$\begin{aligned}\int_0^1 \frac{f\left(\lambda b+(1-\lambda)a+f(\lambda a+(1-\lambda)b\right)}{2}d\lambda &= \frac{1}{2}\left(\int_0^1 f\left(\lambda b+(1-\lambda)a\right)+f(\lambda a+(1-\lambda)b)d\lambda\right)\\ &= \frac{1}{2}\left(\int_0^1 f(\lambda b+(1-\lambda)a)d\lambda+\int_0^1 f(\lambda a+(1-\lambda)b)d\lambda\right)\\ &= \frac{1}{2}\left(\frac{1}{b-a}\int_a^b f(x)dx+\frac{1}{b-a}\int_a^b f(x)dx\right)\\ &= \frac{1}{b-a}\int_a^b f(x)dx\end{aligned}$$

Which therefore proves the inequality (9.11).

9.4.2 Conjuncture

If f is a convex function, are there two real numbers l and L ?, such that the inequality (1.1) is written:

$$f\left(\frac{a+b}{2}\right) \leqslant l \leqslant \frac{1}{b-a}\int_a^b f(x)dx \leqslant L \leqslant \frac{f(a)+f(b)}{2} \tag{9.16}$$

The answer to the conjuncture is affirmative. It can easily be proven by applying the inequality (9.11) to each of the subintervals $[a, \frac{a+b}{2}]$ and $[\frac{a+b}{2}, b]$:

with:

$$l = \frac{1}{2}\left[f\left(\frac{3b+a}{4}\right)+f\left(\frac{b+3a}{4}\right)\right]$$

And

$$L = \frac{1}{2}\left[f\left(\frac{b+a}{2}\right)+\frac{f(a)+f(b)}{2}\right]$$

On the interval $[a, \frac{a+b}{2}]$ the inequality (9.11) becomes:

$$f\left(\frac{3a+b}{4}\right) \leqslant \frac{1}{\frac{a+b}{2}-a}\int_a^{\frac{a+b}{2}} f(x)\,dx \leqslant \frac{f(a)+f\left(\frac{a+b}{2}\right)}{2}$$

On the interval $[\frac{a+b}{2}, b]$ the inequality (9.11) becomes:

$$f\left(\frac{3b+a}{4}\right) \leqslant \frac{1}{b-\frac{a+b}{2}}\int_{\frac{a+b}{2}}^{b} f(x)\,dx \leqslant \frac{f(\frac{a+b}{2})+f(b)}{2}$$

by summing, we obtain:

$$f\left(\frac{3a+b}{4}\right) + f\left(\frac{3b+a}{4}\right) \leqslant \frac{2}{b-a}\int_a^b f(x)\,dx \leqslant \frac{f(a)+f(b)+2f\left(\frac{a+b}{2}\right)}{2},$$

from where

$$\frac{1}{2}\left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{3b+a}{4}\right)\right] \leqslant \frac{1}{b-a}\int_a^b f(x)\,dx \leqslant \frac{1}{2}\left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right]$$
$$l \leqslant \frac{1}{b-a}\int_a^b f(x)\,dx \leqslant L.$$

Thanks to the convexity of f

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &= f\left(\frac{1}{2}(a+b)\right) \\ &= f\left(\frac{1}{2}\left(\frac{3a+b}{4} + \frac{3b+a}{4}\right)\right) \\ &\leqslant \frac{1}{2}\left[\left(\frac{3a+b}{4}\right) + \left(\frac{3b+a}{4}\right)\right] \\ &\leqslant l \end{aligned} \tag{9.17}$$

and,

$$\begin{aligned} \frac{f(a)+f(b)}{2} &= \frac{1}{2}\left[f(a)+f(b)\right] \\ &\geqslant \frac{1}{2}\left[\frac{f(a)+f(b)+2f\left(\frac{a+b}{2}\right)}{2}\right] \\ &\geqslant \frac{1}{2}\left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right] \\ &\geqslant L \end{aligned} \tag{9.18}$$

From (9.17) and (9.18) we obtain the inequality (9.16)

$$f\left(\frac{a+b}{2}\right) \leqslant l \leqslant \frac{1}{b-a}\int_a^b f(x)\,dx \leqslant L \leqslant \frac{f(a)+f(b)}{2}$$

This is what needed to be demonstrated.

9.4.3 Estimation

Theorem 9.22. *Let $f : I \longrightarrow \mathbb{R}$ be a convex function defined on I with a value in $\mathbb{R}$, for all $\lambda \in [0,1]$ and for the two real numbers l and L we have:*

$$f\left(\frac{a+b}{2}\right) \leqslant l(\lambda) \leqslant \frac{1}{b-a}\int_a^b f(x)dx \leqslant L(\lambda) \leqslant \frac{f(a)+f(b)}{2} \tag{9.19}$$

where

$$l(\lambda) = \lambda f\left(\frac{\lambda b+(2-\lambda)a}{2}\right) + (1-\lambda) f\left(\frac{(1+\lambda)b+(1-\lambda)a}{2}\right)$$

and

$$L(\lambda) = \frac{1}{2}\left[f\left(\lambda b+(1-\lambda)a\right) + \lambda f(a) + (1-\lambda)f(b)\right]$$

Proof. We apply the inequality (9.11) on the subinterval $[a, \lambda b+(1-\lambda)a]$ with $\lambda \neq 0$

$$f\left(\frac{a+\lambda b+(1-\lambda)a}{2}\right) \leqslant \frac{1}{\lambda b+(1-\lambda)a-a}\int_a^{\lambda b+(1-\lambda)a} f(x)dx \leqslant \frac{f(a)+f(\lambda b+(1-\lambda)a)}{2}$$

we get

$$f\left(\frac{\lambda b+(2-\lambda)a}{2}\right) \leqslant \frac{1}{\lambda(b-a)}\int_a^{\lambda b+(1-\lambda)a} f(x)dx \leqslant \frac{f(a)+f(\lambda b+(1-\lambda)a)}{2} \tag{9.20}$$

□

We apply the inequality (9.11) again on the subinterval $[\lambda b+(1-\lambda)a, b]$ with $\lambda \neq 1$

$$f\left(\frac{\lambda b+(1-\lambda)a+b}{2}\right) \leqslant \frac{1}{b-(\lambda b+(1-\lambda)a)}\int_{\lambda b+(1-\lambda)a}^{b} f(x)dx \leqslant \frac{f(\lambda b+(1-\lambda)a)+f(b)}{2}$$

We get:

$$f\left(\frac{(1+\lambda)b+(1-\lambda)a}{2}\right)\leqslant\frac{1}{(b-a)(1-\lambda)}\int\limits_{\lambda b+(1-\lambda)a}^{b} f(x)dx\leqslant\frac{f(\lambda b+(1-\lambda)a)+f(b)}{2} \tag{9.21}$$

Multiply (9.20) by λ :

$$\lambda f\left(\frac{\lambda b+(2-\lambda)a}{2}\right)\leqslant\frac{1}{(b-a)}\int\limits_{a}^{\lambda b+(1-\lambda)a} f(x)dx\leqslant\lambda\left(\frac{f(a)+f(\lambda b+(1-\lambda)a)}{2}\right)$$

Multiply (9.21) by $(1-\lambda)$:

$$\begin{aligned}(1-\lambda)f\left(\frac{(1+\lambda)b+(1-\lambda)a}{2}\right)\leqslant&\frac{1}{(b-a)}\int\limits_{\lambda b+(1-\lambda)a}^{b} f(x)dx\\ &\leqslant(1-\lambda)\left(\frac{f(\lambda b+(1-\lambda)a)+f(b)}{2}\right)\end{aligned}$$

By adding the resulting inequalities:

$$\begin{aligned}&\lambda f\left(\frac{\lambda b+(2-\lambda)a}{2}\right)+(1-\lambda)f\left(\frac{(1+\lambda)b+(1-\lambda)a}{2}\right)\\ &\leqslant\frac{1}{b-a}\int\limits_a^b f(x)dx\\ &\leqslant\frac{1}{2}\left(f\left(\lambda b+(1-\lambda)a\right)+\lambda f(a)+(1-\lambda)b\right)\end{aligned}$$

We therefore obtain

$$l(\lambda)\leqslant\frac{1}{(b-a)}\int\limits_a^b f(x)dx\leqslant L(\lambda) \tag{9.22}$$

Thanks to the convexity of f we have:

$$
\begin{aligned}
f\left(\frac{a+b}{2}\right)=&f\left(\lambda\frac{\lambda b+(2-\lambda)a}{2}+(1-\lambda)\frac{(1+\lambda)b+(1-\lambda)a}{2}\right) \\
\leqslant&\lambda f\left(\frac{\lambda b+(2-\lambda)a}{2}\right)+(1-\lambda)f\left(\frac{(1+\lambda))b+(1-\lambda)a}{2}\right) \\
\leqslant&\lambda f\left(\frac{\lambda b+(1-\lambda)a+a}{2}\right)+(1-\lambda)f\left(\frac{\lambda b+(1-\lambda)a+b}{2}\right) \\
\leqslant&\frac{1}{2}\left(f\left(\lambda b+(1-\lambda)a\right)+\lambda f(a)+(1-\lambda)f(b)\right. \\
\leqslant&\frac{f(a)+f(b)}{2}
\end{aligned}
\tag{9.23}
$$

And by (9.22) and (9.23) we obtain the inequality (9.19).

Corollary 9.5. *Let $f : I \longrightarrow \mathbb{R}$ be a convex function on I, for all $\lambda \in [0,1]$ we have the following inequality*

$$
f\left(\frac{a+b}{2}\right)\leqslant \sup_{\lambda\in[0,1]} l(\lambda)\leqslant\frac{1}{b-a}\int_a^b f(x)dx\leqslant\inf_{\lambda\in[0,1]} L(\lambda)\leqslant\frac{f(a)+f(b)}{2}
$$

9.4.4 Application

Let f be a convex function on $I=[a,b]$ we have for $\lambda=\cos^2\theta$, $\theta\in\mathbb{R}$

$$
l(\cos^2\theta)=f\left(\frac{b\cos^2\theta+(1+\sin^2\theta)a}{2}\right)\cos^2\theta+f\left(\frac{(1+\cos^2\theta)b+a\sin^2\theta}{2}\right)\sin^2\theta
$$

and

$$
L(\cos^2\theta)=\frac{1}{2}\left[f\left(b\cos^2\theta+a\sin^2\theta\right)+f(a)\cos^2\theta+f(b)\sin^2\theta\right]
$$

We apply the inequality (9.11) to the subintervals $[a, \cos^2\theta b+(1-\cos^2\theta)\,a]$ and $[\cos^2\theta b+(1-\cos^2\theta)\,a, b]$ respectively, we obtain:

$$
\begin{aligned}
f\left(\frac{b\cos^2\theta+(1+1-\cos^2\theta)a}{2}\right)\leqslant&\frac{1}{\cos^2\theta(b-a)}\int_a^{b\cos^2\theta+(1-\cos^2\theta)a} f(x)dx \\
\leqslant&\frac{f(a)+f(b\cos^2\theta+(1-\cos^2\theta)a)}{2}
\end{aligned}
$$

$$f\left(\frac{(1+\cos^2\theta)b+(1-\cos^2\theta)a}{2}\right)\leqslant\frac{1}{(b-a)(1-\cos^2\theta)}\int\limits_{b\cos^2\theta+(1-\cos^2\theta)a}^{b}f(x)dx$$

$$\leqslant\frac{f(b\cos^2\theta+(1-\cos^2\theta)a)+f(b)}{2}$$

Let us multiply the two results respectively by $\cos^2\theta$ and $(1-\cos^2\theta^2)$.

$$f\left(\frac{b\cos^2\theta+(1+\sin^2\theta)a}{2}\right)\cos^2\theta\leqslant\frac{1}{(b-a)}\int\limits_{a}^{b\cos^2\theta+(1-\cos^2\theta)a}f(x)dx$$

$$\leqslant\frac{f(a)+f(b\cos^2\theta+(\sin^2\theta)a)}{2}\cos^2\theta$$

$$f\left(\frac{(1+\cos^2\theta)b+(1-\cos^2\theta)a}{2}\right)\sin^2\theta\leqslant\frac{1}{(b-a)}\int\limits_{b\cos^2\theta+(1-\cos^2\theta)a}^{b}f(x)dx$$

$$\leqslant\frac{f(b\cos^2\theta+(1-\cos^2\theta)a)+f(b)}{2}\sin^2\theta$$

We sum, we get

$$f\left(\frac{b\cos^2\theta+(1+\sin^2\theta)a}{2}\right)\cos^2\theta+f\left(\frac{(1+\cos^2\theta)b+(\sin^2\theta)a}{2}\right)\sin^2\theta$$

$$\leqslant\frac{1}{(b-a)}\int\limits_a^b f(x)dx$$

$$\leqslant\frac{1}{2}\left[f\left(b\mathrm{cos}^2\theta+a\sin^2\theta\right)+f(a)\mathrm{cos}^2\theta+f(b)\sin^2\theta\right]$$

From where

$$l(\cos^2\theta)\leqslant\frac{1}{(b-a)}\int\limits_a^b f(x)dx\leqslant L(\mathrm{cos}^2\theta)$$

On the other hand

$$
\begin{aligned}
f\left(\frac{a+b}{2}\right) &= f\left(\cos^2\theta \frac{b\cos^2\theta + (2-\cos^2\theta)a}{2} + (1-\cos^2\theta)\frac{(1+\cos^2\theta)b + (1-\cos^2\theta)a}{2}\right) \\
&\leqslant f\left(\frac{b\cos^2\theta + (2-\cos^2\theta)a}{2}\right)\cos^2\theta + f\left(\frac{(1+\cos^2\theta)b + (1-\cos^2\theta)a}{2}\right)(1-\cos^2\theta) \\
&\leqslant f\left(\frac{b\cos^2\theta + (1+\sin^2\theta)a}{2}\right)\cos^2\theta + f\left(\frac{(1+\cos^2\theta)b + a\sin^2\theta}{2}\right)\sin^2\theta \\
&\leqslant \frac{1}{2}\left[f\left(b\cos^2\theta + a\sin^2\theta\right) + f(a)\cos^2\theta + f(b)\sin^2\theta\right] \\
&\leqslant \frac{1}{2}\left[\left(f(a)(\cos^2\theta + \sin^2\theta\right) + f(b)(\cos^2\theta + \sin^2\theta)\right] \\
&\leqslant \frac{f(a)+f(b)}{2}
\end{aligned}
$$

From where

$$
f\left(\frac{a+b}{2}\right) \leqslant l(\cos^2\theta) \leqslant \frac{1}{(b-a)}\int_a^b f(x)dx \leqslant L(\cos^2\theta) \leqslant \frac{f(a)+f(b)}{2}
$$

CHAPTER

10

IMPROPER INTEGRALS

In the previous chapters, all of the functions have been bounded and all of integrals have been computed on closed and bounded intervals. In this section, we relax these restrictions by defining improper Riemann integrals.

If $f : [a, b] \to \mathbb{R}$ be a Riemann integrable function on $[a, b]$, that is, $f \in \Re[a, b]$, then we have

$$\int_a^b f(x)dx = \lim_{c \to a^+} \left(\lim_{d \to b^-} \int_c^d f(x)dx \right) = \lim_{d \to b^-} \left(\lim_{c \to a^+} \int_c^d f(x)dx \right).$$

In fact, define a function $F : [a, b] \to \mathbb{R}$ by

$$F(x) = \int_a^x f(t)dt \text{ for every } x \in [a, b]$$

Then, F is continuous on $[a, b]$ and so

$$\begin{aligned}
\int_a^b f(x)dx &= F(b) - F(a) \\
&= \lim_{d \to b^-} F(d) - \lim_{c \to a^+} F(c) \\
&= \lim_{c \to a^+} \left(\lim_{d \to b^-} (F(d) - F(c)) \right) \\
&= \lim_{c \to a^+} \left(\lim_{d \to b^-} \int_c^d f(x)dx \right).
\end{aligned}$$

Similarly, we have

$$\int_a^b f(x)dx = \lim_{d\to b^-}\left(\lim_{c\to a^+}\int_c^d f(x)dx\right)$$

But, in general, the converse is not true. If the converse is true, then f is said to be improper Riemann integrable on $[a,b]$. This means that if $f:[a,b]\to\mathbb{R}$ is Riemann integrable on $[a,b]$, then f is improper Riemann integrable on $[a,b]$, and so the improper Riemann integral is an extension of the Riemann integral.

Definition 10.1. 1. *Let $a,b\in\mathbb{R}=(-\infty,+\infty)$ with $a<b$ and $f:(a,b]\to\mathbb{R}$ be a function. If $f\in\Re[c,b]$ for every $c\in(a,b)$, then the improper Riemann integral of f on $[a,b]$ is defined by*

$$\int_a^b f(x)dx = \lim_{c\to a^+}\int_c^b f(x)dx$$

provided the limit exists. Then, we say that $\int_a^b f(x)dx$ is convergent. Otherwise, we say that $\int_a^b f(x)dx$ is divergent;

2. *Let $a,b\in\mathbb{R}=(-\infty,+\infty)$ with $a<b$ and $f:[a,b)\to\mathbb{R}$ be a function. If $f\in\Re[a,c]$ for every $c\in(a,b)$, then the improper Riemann integral of f on $[a,b]$ is defined by*

$$\int_a^b f(x)dx = \lim_{c\to b^-}\int_a^c f(x)dx$$

provided the limit exists. Then, we say that $\int_a^b f(x)dx$ is convergent. Otherwise, we say that $\int_a^b f(x)dx$ is divergent;

3. *Let $a,b\in\mathbb{R}=(-\infty,+\infty)$ with $a<b$ and $f:(a,b)\to\mathbb{R}$ be a function. If $f\in\Re[c,d]$ for every $c,d\in(a,b)$ with $c<d$, then the improper Riemann integral of f on $[a,b]$ is defined by*

$$\int_a^b f(x)dx = \lim_{c\to a^+}\left(\lim_{d\to b^-}\int_c^d f(x)dx\right) = \lim_{d\to b^-}\left(\lim_{c\to a^+}\int_c^d f(x)dx\right)$$

provided the limits exist. Then, we say that $\int_a^b f(x)dx$ is convergent. Otherwise, we say that $\int_a^b f(x)dx$ is divergent;

4. *If $c\in(a,b)$ be such that $f\in\Re[a,d]$ and $f\in\Re[e,b]$ for every $d\in(a,c)$ and $e\in(c,b)$, then the improper Riemann integral of f on $[a,b]$ is defined by*

$$\int_a^b f(x)dx = \lim_{d\to c^-}\int_a^d f(x)dx + \lim_{e\to c^+}\int_e^b f(x)dx$$

provided the limits exist, that is, the improper Riemann integrals

$$\int_a^c f(x)dx, \int_c^b f(x)dx$$

are convergent.

Example 10.1. *Let $f : (0, 2] \to \mathbb{R}$ be a function defined by $f(x) = \frac{1}{\sqrt{x}}$ for every $x \in (0, 2]$. Show that f is an improper Riemann integrable function on $[0, 2]$*

Solution. For every $c \in (0, 2]$, since f is continuous on $(0, 2]$, f is Riemann integrable on $[0, 2]$. Further, we have

$$\int_0^2 \frac{1}{\sqrt{x}}dx = \lim_{c\to 0^+} \int_c^2 \frac{1}{\sqrt{x}}dx = \lim_{c\to 0^+} (2\sqrt{2} - 2\sqrt{c}) = 2\sqrt{2}$$

and so f is an improper Riemann integrable function on $[0, 2]$.

Example 10.2. *Let $f : (0, 1] \to \mathbb{R}$ be a function defined by $f(x) = \frac{1}{x}$ for every $x \in (0, 1]$. Show that f is not an improper Riemann integrable function on $[0, 1]$*
Solution. The function f is continuous and Riemann integrable on $(0, 1]$, but $f(x)$ is not defined at $x = 0$. For any $c \in (0, 1]$, we have

$$\int_0^1 \frac{1}{x}dx = [\ln x]_c^1 = \ln 1 - \ln c = -\ln c$$

but the limit

$$\lim_{c\to 0^+} \int_c^1 \frac{1}{x}dx = \lim_{c\to 0^+} (-\ln c) = \infty$$

does not exist. Therefore, $f(x) = \frac{1}{x}$ is not an improper Riemann integrable function on $[0, 1]$.

Definition 10.2.

1. *Let $f : [a, \infty) \to \mathbb{R}$ be a function. If $f \in \Re[a, b]$ for every $a < b$, then the improper Riemann integral of f on $[a, \infty)$ is defined by*

$$\int_a^\infty f(x)dx = \lim_{b\to\infty} \int_a^b f(x)dx$$

provided the limit exists. Then, we say that $\int_a^\infty f(x)dx$ is convergent. Otherwise, we say that $\int_a^\infty f(x)dx$ is divergent;

2. *If $f \in \Re[a, b]$ for every $a < b$, then the improper Riemann integral of f on $(-\infty, b]$ is defined by*

$$\int_{-\infty}^b f(x)dx = \lim_{a\to-\infty} \int_a^b f(x)dx$$

provided the limit exists. Then, we say that $\int_{-\infty}^{b} f(x)dx$ *is convergent. Otherwise, we say that* $\int_{-\infty}^{b} f(x)dx$ *is divergent;*

3. *If* $f : \mathbb{R} \to \mathbb{R}$ *is a function and* $f \in \Re[a, b]$ *for every* $a, b \in \mathbb{R} = (-\infty, \infty)$ *with* $a < b$*, then the improper Riemann integral of* f *on* $(-\infty, \infty)$ *is defined by*

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{a\to-\infty} \int_{a}^{c} f(x)dx + \lim_{b\to\infty} \int_{c}^{b} f(x)dx$$

for every $c \in \mathbb{R}$ *provided the limits exist, that is, the improper Riemann integrals*

$$\int_{c}^{\infty} f(x)dx, \int_{-\infty}^{c} f(x)dx$$

are convergent.

Example 10.3. *Evaluate the following:*

$$\int_{1}^{\infty} \frac{1}{\sqrt{x}}dx$$

Solution. *By the definition of the improper Riemann integral, we have*

$$\int_{1}^{\infty} \frac{1}{\sqrt{x}}dx = \lim_{b\to\infty} \int_{1}^{b} \frac{1}{\sqrt{x}}dx = \lim_{b\to\infty} 2[\sqrt{b} - 1] = \infty$$

Thus, $\int_{1}^{\infty} \frac{1}{\sqrt{x}}dx$ *is divergent.*

Example 10.4. *Evaluate the following:*

$$\int_{-\infty}^{2} \frac{1}{(4-x)^2}dx$$

Solution. *By the definition of the improper Riemann integral, we have*

$$\begin{aligned}
\int_{-\infty}^{2} \frac{1}{(4-x)^2}dx &= \lim_{a\to-\infty} \int_{a}^{2} \frac{1}{(4-x)^2}dx \\
&= \lim_{a\to-\infty} \left[\frac{1}{4-x}\right]_{a}^{2} \\
&= \lim_{a\to-\infty} \left(\frac{1}{2} - \frac{1}{4-a}\right) \\
&= \frac{1}{2} - 0 = \frac{1}{2}
\end{aligned}$$

Thus, $\int_{-\infty}^{2} \frac{1}{(4-x)^2}dx$ *is convergent.*

Example 10.5. *Evaluate the following:*

$$\int_{-\infty}^{\infty} xe^{-x^2}dx$$

Solution. By the definition of the improper Riemann integral, we have

$$\begin{aligned}
\int_{-\infty}^{\infty} xe^{-x^2}dx &= \int_{-\infty}^{0} xe^{-x^2}dx + \int_{0}^{\infty} xe^{-x^2}dx \\
&= \lim_{a\to-\infty}\int_{a}^{0} xe^{-x^2}dx + \lim_{b\to\infty}\int_{0}^{b} xe^{-x^2}dx \\
&= \lim_{a\to-\infty}\left(-\frac{1}{2}+\frac{1}{2}e^{-a^2}\right) + \lim_{b\to\infty}\left(-\frac{1}{2}e^{-b^2}+\frac{1}{2}\right) \\
&= -\frac{1}{2}+\frac{1}{2} = 0.
\end{aligned}$$

Thus, $\int_{-\infty}^{\infty} xe^{-x^2}dx$ is convergent.

Theorem 10.1. *If the improper Riemann integrals*

$$\int_a^{\infty} f(x)dx, \quad \int_a^{\infty} g(x)dx$$

are convergent, then, for every $\alpha, \beta \in \mathbb{R}$, $\int_a^{\infty}(\alpha f(x)+\beta g(x))dx$ is convergent.

Proof. Note that the limit

$$\begin{aligned}
\lim_{b\to\infty}\int_a^b(\alpha f(x)+\beta g(x))dx &= \lim_{b\to\infty}\left(\int_a^b \alpha f(x)dx + \int_a^b \beta g(x)dx\right) \\
&= \alpha\lim_{b\to\infty}\int_a^b f(x)dx + \beta\lim_{b\to\infty}\int_a^b g(x)dx
\end{aligned}$$

exists. Therefore, $\int_a^{\infty}(\alpha f(x)+\beta g(x))dx$ is convergent and

$$\int_a^{\infty}(\alpha f(x)+\beta g(x))dx = \alpha\int_a^{\infty} f(x)dx + \beta\int_a^{\infty} g(x)dx$$

This completes the proof. □

Theorem 10.2. *Let $f : [a,\infty) \to \mathbb{R}$ be a function such that $f(x) \geq 0$ for every $x \in [a,\infty)$ and f is Riemann integrable on $[a,b]$. Then, the following are equivalent:*

(1) The improper Riemann integral $\int_a^{\infty} f(x)dx$ is convergent;

(2) The set

$$\left\{\int_a^b f(x)dx : b \in (a,\infty)\right\}$$

is bounded and, further,

$$\int_a^{\infty} f(x)dx = \sup\left\{\int_a^b f(x)dx : b \in (a,\infty)\right\}$$

Proof. (1) $\Longrightarrow$ (2) Define a function $F : [a, \infty) \to \mathbb{R}$ by

$$F(b) = \int_a^b f(x)dx$$

and suppose that the improper Riemann integral $\int_a^\infty f(x)dx$ is convergent. Since F is increasing, for every $b \in [a, \infty)$, we have

$$F(b) = \int_a^b f(x)dx \leq \int_a^\infty f(x)dx$$

and so

$$\left\{\int_a^b f(x)dx : b \in (a, \infty)\right\}$$

is bounded.

(2) $\Longrightarrow$ (1) Suppose that

$$\left\{\int_a^b f(x)dx : b \in (a, \infty)\right\}$$

is bounded. Let $L = \sup\{F(b) : b \in (a, \infty)\}$. Then, for every $\varepsilon > 0$, there exists a number $b \in \mathbb{R}$ such that $L - \varepsilon < F(b)$. Let $x \geq b$. Since $f(x) \geq 0$, we have

$$L - \varepsilon < F(b) \leq F(x) \leq L < L + \varepsilon$$

Since F is increasing and bounded from above, we have

$$\lim_{b\to\infty} F(b) = \sup\{F(b) : b \in (a, \infty)\}$$

and

$$\int_a^\infty f(x)dx = \sup\left\{\int_a^b f(x)dx : b \in (a, \infty)\right\}$$

This completes the proof. $\square$

Theorem 10.3. *Let $f, g : [a, \infty) \to \mathbb{R}$ be two functions such that, for every $x \in [a, \infty), 0 \leq f(x) \leq g(x)$ and $\int_a^\infty g(x)dx$ is convergent. Then, $\int_a^\infty f(x)dx$ is convergent and*

$$\int_a^\infty f(x)dx \leq \int_a^\infty g(x)dx$$

Proof. Let $b \in [a, \infty)$. Then, we have

$$\int_a^b f(x)dx \leq \int_a^b g(x)dx \leq \int_a^\infty g(x)dx$$

Since $\int_a^\infty g(x)dx$ is convergent, it follows that $\int_a^\infty f(x)dx$ is convergent and

$$\int_a^\infty f(x)dx \leq \int_a^\infty g(x)dx$$

This completes the proof. $\square$

Example 10.6. *Show that $\int_1^\infty \sqrt{x}e^{-x^2}dx$ is convergent.*

Solution. For every $x \geq 1$, we have

$$0 \leq \sqrt{x}e^{-x^2} \leq xe^{-x^2}$$

and

$$\int_1^\infty xe^{-x^2}dx = \lim_{b\to\infty}\int_1^b xe^{-x^2}dx = \lim_{b\to\infty}\left(-\frac{1}{2}e^{-b^2} + \frac{1}{2e}\right) = \frac{1}{2e}$$

Therefore, by Theorem 8.5.2, it follows that $\int_1^\infty \sqrt{x}e^{-x^2}dx$ is convergent.

Theorem 10.4. *Let $f : [a, \infty) \to \mathbb{R}$ be a function such that, for every $b \in [a, \infty)$, f is Riemann integrable on $[a, b]$ and $\int_a^\infty |f(x)|dx$ is convergent. Then, $\int_a^\infty f(x)dx$ is convergent and*

$$\left|\int_a^\infty f(x)dx\right| \leq \int_a^\infty |f(x)|dx$$

Proof. For every $x \in [a, \infty)$, we have

$$-|f(x)| \leq f(x) \leq |f(x)|$$

and so

$$0 \leq f(x) + |f(x)| \leq 2|f(x)|$$

Thus, for every $b \in [a, \infty)$, $f + |f|$ is Riemann integrable on $[a, b]$. Since $\int_a^\infty 2|f(x)|dx$ is convergent, then, $\int_a^\infty (f(x) + |f(x)|)dx$ is convergent. On the other hand, for every $x \in [a, \infty)$, since we have

$$f(x) = f(x) + |f(x)| - |f(x)|$$

it follows that $\int_a^\infty f(x)dx$ is convergent.

Now, let $\int_a^\infty |f(x)|dx = L$. Then, since $-|f(x)| \leq f(x) \leq |f(x)|$ for every $x \in [a, \infty)$, we have

$$-L = -\int_a^\infty |f(x)|dx \leq \int_a^\infty f(x)dx \leq \int_a^\infty |f(x)|dx = L$$

and so

$$\left|\int_a^\infty f(x)dx\right| \leq L = \int_a^\infty |f(x)|dx$$

This completes the proof. $\square$

Definition 10.3. 1. *An improper integral $\int_a^b f(x)dx$ is said to be absolutely convergent if the improper integral $\int_a^b |f(x)|dx$ converges;*

2. *An improper integral $\int_a^b f(x)dx$ is said to be conditionally convergent if $\int_a^b f(x)dx$ converges, but $\int_a^b |f(x)|dx$ diverges.*

The following exercise is a theorem called Bertrand Integrals

Exercise 10.1. *Discuss the convergence or divergence of the Bertrand Integrals*

$$\int_2^\infty \frac{1}{x^\alpha \ln^\beta(x)}dx$$

depending on the parameters α and β.

Solution

Set $\mu = \frac{1+\alpha}{2}$. If $\alpha > 1$, then $1 < \mu < \alpha$. Since $\lim_{x\to\infty} \frac{x^\mu}{x^\alpha \ln^\beta(x)} = 0$, for any $\beta \in \mathbb{R}$, there exists $A > 0$ such that for any $x \geq A$, we have $\frac{x^\mu}{x^\alpha \ln^\beta(x)} \leq 1$ which implies $\frac{1}{x^\alpha \ln^\beta(x)} \leq \frac{1}{x^\mu}$. Since the improper integral $\int_1^\infty \frac{1}{x^\mu}dx$ is convergent, the basic comparison test will force $\int_2^\infty \frac{1}{x^\alpha \ln^\beta(x)}dx$ to be convergent. If $\alpha < 1$, then $1 > \mu > \alpha$. Since $\lim_{x\to\infty} \frac{x^\mu}{x^\alpha \ln^\beta(x)} = \infty$, for any $\beta \in \mathbb{R}$, there exists $A > 0$ such that for any $x \geq A$, we have $\frac{x^\mu}{x^\alpha \ln^\beta(x)} \geq 1$ which implies $\frac{1}{x^\alpha \ln^\beta(x)} \geq \frac{1}{x^\mu}$. Since the improper integral $\int_1^\infty \frac{1}{x^\mu}dx$ is divergent, the basic comparison test will force $\int_2^\infty \frac{1}{x^\alpha \ln^\beta(x)}dx$ to be divergent. Finally, assume $\alpha = 1$. Then for any $A > 2$, we have

$$\int_2^A \frac{1}{x\ln^\beta(x)}dx = \int_{\ln(2)}^{\ln(A)} \frac{1}{x^\beta}dx$$

Since $\int_{\ln(2)}^\infty \frac{1}{x^\beta}dx$ is convergent if and only if $\beta > 1$, $\int_2^\infty \frac{1}{x\ln^\beta(x)}dx$ is convergent if and only if $\beta > 1$.

The following exercise is a theorem called Riemann Integrals

Exercise 10.2. *Let α be a real number and set*

$$I(\alpha) = \int_0^1 \frac{dx}{x^\alpha} \text{ and } J(\alpha) = \int_1^{+\infty} \frac{dx}{x^\alpha}$$

We have

$$\lim_{\varepsilon\to 0^+}\int_\varepsilon^1 \frac{dx}{x^\alpha} = \lim_{\varepsilon\to 0^+}\begin{cases} -\ln(\varepsilon), & \text{if } \alpha = 1, \\ \frac{1}{1-\alpha}\left(1-\frac{1}{\varepsilon^{\alpha-1}}\right), & \text{if } \alpha \neq 1 \end{cases} = \begin{cases} +\infty, & \text{if } \alpha \geq 1 \\ \frac{1}{1-\alpha}, & \text{if } \alpha < 1 \end{cases}$$

and

$$\lim_{A\to +\infty}\int_1^A \frac{dx}{x^\alpha} = \lim_{\varepsilon\to 0^+}\begin{cases} \ln(A), & \text{if } \alpha = 1, \\ \frac{1}{1-\alpha}\left(\frac{1}{A^{\alpha-1}}-1\right), & \text{if } \alpha \neq 1 \end{cases} = \begin{cases} +\infty, & \text{if } \alpha \leq 1 \\ \frac{1}{1-\alpha}, & \text{if } \alpha > 1 \end{cases}$$

Therefore

$$I(\alpha) \text{ converges } \Leftrightarrow \alpha < 1,$$
$$J(\alpha) \text{ converges } \Leftrightarrow \alpha > 1.$$

The integrals $I(\alpha)$ and $J(\alpha)$ are commonly called Riemann integrals.

Theorem 10.5 (Cauchy Criterion)**.** *Let $f : [a,\infty) \to \mathbb{R}$ be Riemann integrable on bounded intervals. Then $\int_a^\infty f(x)dx$ converges if and only if for every $\varepsilon > 0$ there exists $A > a$ such that for any $t_1, t_2 > A$ we have*

$$\left|\int_{t_1}^{t_2} f(x)dx\right| < \varepsilon$$

Proof. Set $F(t) = \int_a^t f(x)dx$. It is clear that the improper integral $\int_a^\infty f(x)dx$ converges if and only if $\lim_{t\to\infty} F(t)$ exists. Assume that the improper integral converges. Then for any $\varepsilon > 0$, there exists $A > 0$ such that for any $t > A$ we have

$$\left|F(t) - \int_a^\infty f(x)dx\right| < \frac{\varepsilon}{2}$$

Hence for any $t_1, t_2 > A$, we have

$$|F(t_1) - F(t_2)| \leq \left|F(t_1) - \int_a^\infty f(x)dx\right| + \left|F(t_2) - \int_a^\infty f(x)dx\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Assume the converse is true, i.e., for every $\varepsilon > 0$ there exists $A > a$ such that for any $t_1, t_2 > A$ we have

$$\left|\int_{t_1}^{t_2} f(x)dx\right| < \varepsilon$$

Let us prove that the improper integral $\int_a^\infty f(x)dx$ is convergent. Note that $\lim_{t\to\infty} F(t)$ exists if and only if for any sequence $\{t_n\}$ which goes to ∞, the sequence $\{F(t_n)\}$ is convergent. In $\mathbb{R}$ convergence of sequences is equivalent to the Cauchy behavior. Hence

$\lim_{t\to\infty} F(t)$ exists if and only if for any sequence $\{t_n\}$ which goes to ∞, the sequence $\{F(t_n)\}$ is Cauchy. Our assumption forces this to be true. Indeed, let $\{t_n\}$ be a sequence which goes to 0 . Let $\varepsilon > 0$. Then there exists $A > 0$ such that for any $t_1, t_2 > A$ we have

$$|F(t_1) - F(t_2)| = \left|\int_{t_1}^{t_2} f(x)dx\right| < \varepsilon.$$

Since $\{t_n\}$ goes to ∞, there exists $n_0 \geq 1$ such that for any $n \geq n_0$ we have $t_n > A$. So for any $n, m \geq n_0$, we have

$$|F(t_n) - F(t_m)| = \left|\int_{t_n}^{t_m} f(x)dx\right| < \varepsilon$$

which translates into $\{F(t_n)\}$ being a Cauchy sequence. □

Theorem 10.6 (Abel's test). *Assume that the functions f and g defined on $[a,\infty)$ satisfy the following conditions:*

(a) g is monotone and bounded on $[a,\infty)$,

(b) the improper integral $\int_a^\infty f(x)dx$ is convergent. Then $\int_a^\infty f(x)g(x)dx$ is also convergent.

Proof. Let us use the Cauchy criteria proved in the previous problem to prove our claim. Let $\varepsilon > 0$. Since $g(x)$ is bounded, there exists $M > 0$ such that $|g(x)| \leq M$ for all $x \in [a,\infty)$. Since $\int_a^\infty f(x)dx$ is convergent, there exists $A > 0$ such that for any $t_1, t_2 > A$ we have

$$\left|\int_{t_1}^{t_2} f(x)dx\right| < \frac{\varepsilon}{2M}$$

By the Second Mean Value Theorem for integrals, and for any $t_1, t_2 > A$, there exists c between t_1 and t_2 such that

$$\int_{t_1}^{t_2} f(x)g(x)dx = g(t_1)\int_{t_1}^{c} f(x)dx + g(t_2)\int_{c}^{t_2} f(x)dx$$

Hence

$$\left|\int_{t_1}^{t_2} f(x)g(x)dx\right| \leq |g(t_1)|\left|\int_{t_1}^{c} f(x)dx\right| + |g(t_2)|\left|\int_{c}^{t_2} f(x)dx\right|.$$

But, $\left|\int_{t_1}^{c} f(x)dx\right| < \frac{\varepsilon}{2M}$ and $\left|\int_{c}^{t_2} f(x)dx\right| < \frac{\varepsilon}{2M}$, which forces the inequality $\left|\int_{t_1}^{t_2} f(x)g(x)dx\right| < \frac{\varepsilon}{2M}|g(t_1)| + \frac{\varepsilon}{2M}|g(t_2)| < \varepsilon$ to be true. □

Theorem 10.7. *(Dirichlet's test.) Let functions f, and g satisfy the following properties:*

1. *There exists $C \in \mathbb{R}$ such that $\int_a^z f(x)dx \leqslant C, \quad \forall z \geq a$.*
2. *The function g is monotone on $[a,+\infty)$ and $g(x) \to 0, x \to +\infty$.*
   Then the integral $\int_a^\infty f(x)g(x)dx$ converges.

CHAPTER

11

FIRST ORDER DIFFERENTIAL EQUATIONS

Introduction

The laws of the universe are written in the language of mathematics. Algebra is sufficient to solve many static problems, but the most interesting natural phenomena involve change and are described by equations that relate changing quantities. Because the derivative $\frac{dy(x)}{dx} = y'(x)$ of the function y is the rate at which the quantity $y(x)$ is changing with respect to the independent variable x, it is natural that equations involving derivatives are frequently used to describe the changing universe.

11.1 Notations and definitions

Definition 11.1. *An equation relating an unknown function and one or more of its derivatives is called a differential equation,*

$$F\left(y, y', y'', \ldots, y^{(n)}\right) = 0$$

Definition 11.2. *The order of a differential equation is the highest order of the derivative appearing in the differential equation. The degree of a differential equation is the* *highest power of the highest order derivative in a differential equation. The degree of the differential equation is always a positive integer.*

Definition 11.3. *We call solution or integral of a differential equation of order n on a certain open interval I of $\mathbb{R}$, any function y defined on this interval,*

$$\begin{aligned} y: \quad & I \to \mathbb{R} \\ & x \mapsto y(x) \end{aligned}$$

such that y is n-times differentiable at any point of I and satisfies this differential equation.

Definition 11.4. *A linear differential equation of order n is any equation that can be expressed in the form*

$$a_0(x)y^{(n)}(x) + a_1(x)y^{(n-1)}(x) + \ldots + a_n(x)y(x) = f(x)$$

where f and a_i are specified functions for $i = 0, \ldots, n$ and $a_0 \neq 0$. a_i are called the coefficients of the DE.

Definition 11.5. *If y is a function of a single variable, the equation is called an ordinary differential equation (ODE).*

Examples 11.1.

1. $y' \ln x + x^2 y + \cos x = 0$ *linear differential equation of order 1 and degree 1.*
2. $8y'' + y' - 3y = xe^x \sin x$ *linear differential equation of order 2 and degree 1.*
3. $(y'')^3 + (y')^2 = -1$ *nonlinear differential equation of order 2 and degree 3.*
4. $y^{(4)} + 5yy'' + y = 3$ *nonlinear differential equation of order 4, and degree1*

11.2 First order differential equations

First order linear differential equations with separable variables

Definition 11.6. *We call linear differential equation of order 1 with separable variables, any equation of the form*

$$y' = f(x)h(y)$$

where f and h are functions of class C^1 on interval I of $\mathbb{R}$.

Resolution

We can reduce this equation to a linear differential equation of order 1 called with separated variables of the form

$$f(x)dx = g(y)dy$$

where $g(y) = \frac{1}{h(y)}, \forall y \in I$, such that $h(y) \neq 0$, then we integrate the two sides each with respect to its variable.

Example 11.1.

Solve (integrate) the following differential equations

1. $y' - x^2 y = x^2$
2. $y'(x^2 - 3) + 2xy = 0.$
3. $y'(x^2 + 1) = \sqrt{1 - y^2}.$

Solution

1.

$$y' - x^2 y = x^2 \Leftrightarrow \frac{dy}{dx} = (1+y)x^2$$

separating the variables and assuming that $y \neq -1$, we have

$$\frac{dy}{1+y} = x^2 dx$$

hence by integrating the left side with respect to y and the right side with respect to x, we obtain

$$\begin{aligned} \ln|1+y| &= \frac{1}{3}x^3 + c, c \in \mathbb{R} \\ &\Rightarrow |1+y| = e^{\frac{1}{3}x^3 + c} \\ &\Rightarrow y = e^{\frac{1}{3}x^3 + c} - 1 \end{aligned}$$

then setting $k = \pm e^c$ we have

$$y(x) = ke^{\frac{1}{3}x^3} - 1, k \in \mathbb{R}$$

2.

$$y'\left(x^2 - 3\right) - 2xy = 0 \Leftrightarrow \frac{dy}{dx}\left(x^2 - 3\right) = 2xy$$

assuming that $x \neq -\sqrt{3}$ and $x \neq \sqrt{3}$, we have

$$\frac{dy}{y} = \frac{2x}{x^2-3}dx$$

from where

$$\ln|y| = \ln\left|x^2-3\right| + c, c \in \mathbb{R},$$

then setting $k = \pm e^c$ we have

$$y(x) = k\left(x^2-3\right), k \in \mathbb{R}.$$

3.

$$y'\left(x^2+1\right) = \sqrt{1-y^2} \Leftrightarrow \frac{dy}{dx}\left(x^2+1\right) = \sqrt{1-y^2}$$

assuming that $y \neq -1$ and $y \neq 1$, we have

$$\frac{dy}{\sqrt{1-y^2}} = \frac{dx}{x^2+1}$$

from where

$$\arcsin y = \arctan x + k, k \in \mathbb{R}$$

as a result

$$y(x) = \sin(\arctan x + k), k \in \mathbb{R}.$$

Remark 11.1. *To determine the constant k it suffices to give an initial condition,* $y_0 = y\left(x_0\right)$.

First order homogeneous linear differential equations

Definition 11.7.

A first order homogeneous linear differential equation is of the form

$$y'(x) + a(x)y(x) = 0$$

where a(x) is a continuous function on an interval I of $\mathbb{R}$.
Solving this DE consists in separating the variables such that

$$\frac{dy}{dx} + a(x)y = 0 \Leftrightarrow \frac{dy}{y} = -a(x)dx$$

whence by integrating

$$\ln|y| = -\int a(x)dx + c, c \in \mathbb{R}$$

hence the solution of DE is said to be homogeneous solution and it is given by

$$y_{\text{hom}}(x) = ke^{-\int a(x)dx}, \text{ where } k = e^c.$$

Remark 11.2. *1- For a homogeneous equation, the trivial solution $y = 0$ is a solution.*
2- The solution here is not unique, but if we also have a particular solution y_p for the initial condition $x_0 \in I$, such that $y_p = y(x_0)$, then we can calculate the constant k and in this case, the DE will have a unique solution.

Example 11.2. *Solve the homogeneous differential equation $3y' + e^x y = 0$.*

$$3y' + e^x y = 0 \Leftrightarrow 3\frac{dy}{dx} = -e^x y \Leftrightarrow \frac{dy}{y} = -\frac{1}{3}e^x dx,$$

whence by integrating the two members

$$\ln|y| = -\frac{1}{3}\int e^x dx + c, c \in \mathbb{R}$$

as a result

$$\ln|y| = -\frac{1}{3}e^x + c$$

so the homogeneous solution is given by

$$y_{hom}(x) = ke^{-\frac{1}{3}e^x}, \ ork = \pm e^c$$

such that $y = 0$ is a trivial solution.

First order nonhomogeneous linear differential equations

Definition 11.8. *A first order nonhomogenuous or with second member linear differential equation is of the form*

$$a(x)y'(x) + b(x)y(x) = f(x)$$

where a, b and f are given functions, continuous on an interval I of $\mathbb{R}$, not identically null on I.

Resolution method

Step 1 First of all we solve the associated homogeneous equation (Eq.Hom)

$$\text{Eq. Hom: } a(x)y'(x)+b(x)y(x)=0$$

which is an equation with separable variables

$$a(x)y'+b(x)y=0 \Leftrightarrow \frac{y'}{y}=-\frac{a(x)}{b(x)} \Leftrightarrow \frac{dy}{y}=-\frac{a(x)}{b(x)}dx$$

whence by integrating we have

$$\ln|y|=-\int \frac{a(x)}{b(x)}dx+c, c\in\mathbb{R}$$

hence the homogeneous solution of the homogeneous equation is given by

$$\begin{aligned} &|y_{\text{hom}}(x)|=e^{-\int \frac{a(x)}{b(x)}dx+c} \\ &\Leftrightarrow y_{\text{hom}}(x)=ke^{-\int \frac{a(x)}{b(x)}dx}, \text{ where } k=\pm e^c\in\mathbb{R}. \end{aligned}$$

Step 2 To have the general solution of the differential equation, we distinguish two cases:

Case 1 If we know a particular solution y_p of the inhomogeneous differential equation, then we give the general solution by the formula

$$y_{\text{gle}}=y_{\text{hom}}+y_p$$

Case 2 If we do not know any particular solution of the inhomogeneous differential equation, then we proceed by the method of the variation of the constant, i.e. replace the homogeneous solution y_{Hom} in the non-homogeneous differential equation by considering the constant k as a function of the variable x.

Indeed, let y_p be a particular solution of the inhomogeneous differential equation, and y_{gle} a general solution of the inhomogeneous differential equation, then $y_{gle}-y_p$ is a solution of the homogeneous equation, indeed,

y_p verifies the inhomogeneous differential equation then

$$a(x)y_p'(x)+b(x)y_p(x)=f(x)$$

and y_{gle} also satisfies the inhomogeneous differential equation then

$$a(x)y_{gle}'(x)+b(x)y_{gle}(x)=f(x)$$

and calculating the difference, we have

$$a(x)\left(y_{gle}(x)-y_p(x)\right)'+b(x)\left(y_{gle}(x)-y_p(x)\right)=0$$

hence $y_{gle}-y_p$ verifies the homogeneous equation, thus

$$y_{\text{gle}}-y_p=y_{\text{hom}}\Leftrightarrow y_{\text{gle}}=y_{\text{hom}}+y_p$$

Example 11.3. *Solve*

$$y'\cos x+y\sin x=1$$

Homogeneous equation: We start by writing the homogeneous equation of the nonhomogeneous differential equation in the form

$$\textit{Eq. Hom: } y'\cos x+y\sin x=0$$

It is an equation with separable variables

$$\Leftrightarrow\frac{y'}{y}=-\frac{\sin x}{\cos x}\Leftrightarrow\frac{1}{y}dy=-\frac{\sin x}{\cos x}dx$$

whence by integrating we have
$\ln|y|=-\int\frac{\sin x}{\cos x}dx+c, c\in\mathbb{R}\Rightarrow\ln|y|=\ln|\cos x|+c\Rightarrow y=k\cos x$, *where* $k=\pm e^c$.
Therefore, the homogeneous solution is given by

$$y_{hom}(x)=k\cos x,\ \textit{where } k\in\mathbb{R}.$$

We then notice that this differential equation admits as an obvious particular solution the function $y_p=\sin x$, *indeed* $y'=\cos x$ *from where* y_p *verifies the nonhomogeneous differential equation, so we can use case 1 and we have*

$$y_{gle}(x)=k\cos x+\sin x,k\in\mathbb{R}$$

Remark 11.3. *If we do not win to find a particular solution of the NDE, we can use the method of the variation of the constant* k *in the homogeneous solution* $y_{Hom}(x)=k\cos x$. *Indeed,*

$$y_{hom}(x)=k(x)\cos x\Rightarrow y'(x)=k'\cos x-k\sin x$$

then substituting in the NDE, we get

$$k'\cos^2 x=1\Leftrightarrow dk=\frac{dx}{\cos^2 x}$$

and by integrating we have

$$k(x) = \tan x + c, c \in \mathbb{R}.$$

Then we replace in the Homogeneous Differential equation, then we get the general solution

$$y_{gle}(x) = (\tan x + c) \cos x$$

$$\text{hence } y_{gle}(x) = \sin x + c \cos x, c \in \mathbb{R}.$$

Example 11.4. *Solve the differential equation*

$$y' + y = e^{-x}$$

with the particular solution $y(0) = 1$.

Solution

$$\textit{Eq. Hom: } y' + y = 0 \Leftrightarrow \frac{dy}{dx} = -y \Leftrightarrow \frac{dy}{y} = -dx$$

then by integrating on both sides we get

$$\ln |y| = -x + c_1, c_1 \in \mathbb{R}$$

hence by setting $k = \pm e^{c_1}$

$$y_{hom}(x) = ke^{-x}, \text{ where } k \in \mathbb{R}.$$

We vary the constant k, we then have

$$y'(x) = k'e^{-x} - ke^{-x}$$

then we substitute in NDE to obtain

$$k' = 1 \Leftrightarrow dk = dx$$

whence by integrating we have

$$k(x) = x + c, c \in \mathbb{R}$$

as a result

$$y_{gle}(x) = (x + c)e^{-x}, c \in \mathbb{R}$$

as $y(0) = 1$ then, $c = 1$ so

$$y(x) = (x+1)e^{-x}$$

Example 11.5.

Solve the following differential equation

$$\left(1+x^2\right) y' - \frac{y}{\arctan x} = \arctan x$$

$$\text{Eq. Hom } : \left(1+x^2\right) y' - \frac{y}{\arctan x} = 0,$$

$$\left(1+x^2\right) y' - \frac{y}{\arctan x} = 0 \Leftrightarrow \frac{dy}{y} = \frac{dx}{(1+x^2)\arctan x}$$

By integrating both sides, we get

$$\ln|y| = \int \frac{1}{(1+x^2)\arctan x} dx + c_1, c_1 \in \mathbb{R}$$

$$CV : t = \arctan x \Rightarrow dt = \frac{dx}{1+x^2}$$

from where

$$\ln|y| = \int \frac{dt}{t} + c_1 = \ln|t| + c_1, c_1 \in \mathbb{R}$$

SO

$$\ln|y| = \ln|\arctan x| + c_1, c_1 \in \mathbb{R},$$

so by setting $k = \pm e^{c_1}$

$$y_{\text{hom}}(x) = k \arctan x, \text{or } k \in \mathbb{R}.$$

By the technique of variantion of the constant, we suppose temporally that k, isn't a constant, thus

$$y'(x) = k' \arctan x + \frac{k}{1+x^2},$$

than we replace in the NDE

$$\left(1+x^2\right)\left[k' \arctan x + \frac{k}{1+x^2}\right] - k = \arctan x.$$

so

$$k'\left(1+x^2\right) = 1 \Leftrightarrow k' = \frac{1}{1+x^2} \Leftrightarrow dk = \frac{dx}{1+x^2}$$

by integration, we get

$$k(x) = \arctan x + c, c \in \mathbb{R},$$

then

$$y_{gle}(x) = \arctan^2 x + c \arctan x, c \in \mathbb{R}.$$

11.3 Bernoulli Differential Equations

Definition 11.9. *Differential equations in this form*

$$y'(x) + a(x)y(x) + b(x)y^{\alpha}(x) = 0$$

where $\alpha \in \mathbb{R}^*, \alpha \neq 1$ *and* a, b *are two given functions, of class* C^0 *on an interval* I *in* $\mathbb{R}$, *are called Bernoulli Equations.*

Resolution method

First, we assume that $y \neq 0$ because we are looking for a non-trivial solution.

The technique consists in dividing the DE by $y^{\alpha}(x)$, which leads us after an adequate change of variables to a linear differential equation of order 1. Indeed, we have

$$\frac{y'(x)}{y^{\alpha}(x)} + \frac{a(x)}{y^{\alpha-1}(x)} + b(x) = 0$$

by CV : $z(x) = y^{1-\alpha}(x) = \frac{1}{y^{\alpha-1}(x)}$, we get

$$z'(x) = (1-\alpha)y^{-\alpha}(x)y'(x) = (1-\alpha)\frac{y'(x)}{y^{\alpha}(x)}$$

so $\frac{y'(x)}{y^{\alpha}(x)} = \frac{1}{1-\alpha} z'(x)$, hence

$$\frac{1}{(1-\alpha)} z'(x) + a(x)z(x) + b(x) = 0$$

which is a linear differential equation of order 1 that we know how to deal with.

Example 11.6.

Integrate the following differential equation:

$$y' + xy = xy^2$$

It is of Bernoulli's form with $\alpha = 2$.

We assume that $y(x) \neq 0$ and we divide by y^2

$$\frac{y'}{y^2} + \frac{x}{y} = x$$

$$CV : z(x) = y^{-1}(x) = \frac{1}{y(x)}, \text{ hence } z'(x) = \frac{-y'(x)}{y^2(x)},$$

then we substitute in DE to find

$$-z' + xz = x$$

which is a linear differential equation of order 1.

Note that this equation can be solved by the method of separation of variables or by the method of variation of the constant.

Separation of variables method

$$-z' + xz = x \Leftrightarrow z' = x(z-1) \Leftrightarrow \frac{dz}{z-1} = xdx$$

and integrating on each side we have

$$\begin{aligned} &\ln|z-1| = \frac{x^2}{2} + c, c \in \mathbb{R}, \\ &\Rightarrow |z-1| = e^{\frac{x^2}{2}+c} = e^c \cdot e^{\frac{x^2}{2}}, c \in \mathbb{R}, \\ &\Rightarrow z - 1 = ke^{\frac{x^2}{2}} \text{ with } k = \pm e^c, \end{aligned}$$

from where

$$z_{gle}(x) = 1 + ke^{\frac{x^2}{2}}, \text{ with } k = \pm e^c \in \mathbb{R}$$

and since $y(x) = \frac{1}{z(x)}$, then

$$y_{gle}(x) = \frac{1}{1 + ke^{\frac{x^2}{2}}}, k \in \mathbb{R}.$$

For more explanations, we will calculate z a second time by the method of the variation of the constant

Constant variation method

We first solve the homogeneous equation

$$-z' + xz = 0 \Leftrightarrow \frac{dz}{z} = x$$

from where

$$
\begin{aligned}
&\ln |z| = \frac{1}{2}x^2 + c_1, c_1 \in \mathbb{R} \\
&\Leftrightarrow |z| = e^{\frac{1}{2}x^2 + c_1} = e^{c_1} \cdot e^{\frac{1}{2}x^2} \\
&\Leftrightarrow z = \pm e^{c_1} \cdot e^{\frac{1}{2}x^2} = ke^{\frac{1}{2}x^2} \text{ with } k = \pm e^{c_1}.
\end{aligned}
$$

SO

$$z_{\text{hom}} = ke^{\frac{1}{2}x^2}.$$

We vary the constant k, we then have

$$z' = k'e^{\frac{1}{2}x^2} + kxe^{\frac{1}{2}x^2}$$

then we substitute in NDE to obtain

$$-k'e^{\frac{1}{2}x^2} - kxe^{\frac{1}{2}x^2} + kxe^{\frac{1}{2}x^2} = x$$

from where

$$k' = -xe^{-\frac{1}{2}x^2} \Leftrightarrow dk = -xe^{-\frac{1}{2}x^2}dx$$

We integrate both sides and we get

$$k(x) = e^{-\frac{1}{2}x^2} + K, K \in \mathbb{R},$$

then

$$z_{gle}(x) = \left(e^{-\frac{1}{2}x^2} + K\right) e^{\frac{1}{2}x^2}$$

or, equivalently

$$z_{\text{gle}}\left(x\right) = 1 + Ke^{\frac{1}{2}x^2}, K \in \mathbb{R}.$$

Example 11.7. *Integrate the following differential equation*

$$y' + \frac{2}{x}y = \frac{e^x}{\sqrt{y}}$$

Solution It is an equation of Bernoulli form with $\alpha = -\frac{1}{2}$. We assume that $y(x) \neq 0$ and we divide by $y^{-\frac{1}{2}}$

$$y'\sqrt{y} + \frac{2}{x}y^{\frac{3}{2}} = e^x$$

$$CV : z(x) = y^{\frac{3}{2}}(x), \text{ hence } z'(x) = \frac{3}{2}y'(x)y^{\frac{1}{2}}(x),$$

then we replace in the concerned equation, we get

$$\frac{2}{3}z' + \frac{2}{x}z = e^x$$

which is a linear differential equation of order 1.
Homogeneous equation

$$\frac{2}{3}z' + \frac{2}{x}z = 0$$
$$\Leftrightarrow \frac{z'}{z} = -\frac{3}{x} \Leftrightarrow \frac{dz}{z} = -\frac{3}{x}dx$$

by integrating we get

$$\begin{aligned} &\ln|z| = -3\ln|x| + c_1, c_1 \in \mathbb{R}, \\ &\Leftrightarrow |z| = e^{-3\ln|x|+c_1} = e^{c_1}e^{-3\ln|x|} \\ &\Leftrightarrow |z| = \frac{e^{c_1}}{x^3} \end{aligned}$$

hence the homogeneous solution

$$z_{\text{hom}}(x) = \frac{k}{x^3}, \text{ where } k = \pm e^{c_1} \in \mathbb{R}$$

Variation of the constant
We vary the constant k, then

$$z'(x) = \frac{k'(x)}{x^3} - 3\frac{k(x)}{x^4}$$

and we substitute in Differential equation, which gives

$$\frac{2}{3}\left(\frac{k'(x)}{x^3} - 3\frac{k(x)}{x^4}\right) + \frac{2k(x)}{x^4} = e^x \Leftrightarrow \frac{2}{3}\frac{k'(x)}{x^3} = e^x$$
$$\Leftrightarrow dk = \frac{3}{2}x^3e^x dx \Rightarrow k(x) = \frac{3}{2}\int x^3e^x dx$$

then we integrate by parts 3 times
IPP 1 : $\begin{cases} u = x^3 \\ dv = e^x dx \end{cases} \Rightarrow \begin{cases} du = 3x^2 dx \\ v = e^x \end{cases}$
from where

$$k(x) = \frac{3}{2}\left[x^3e^x - 3\int x^2e^x dx\right]$$

IPP 2 : $\begin{cases} u = x^2 \\ dv = e^x dx \end{cases} \Rightarrow \begin{cases} du = 2x dx \\ v = e^x \end{cases}$ *hence*

$$k(x) = \frac{3}{2}x^3e^x - \frac{9}{2}\left[x^2e^x - 2\int xe^x dx\right]$$

$$IPP\ 3 : \begin{cases} u = x \\ dv = e^x dx \end{cases} \Rightarrow \begin{cases} du = dx \\ v = e^x \end{cases}$$

from where

$$\begin{aligned} k(x) &= \frac{3}{2}x^3e^x - \frac{9}{2}\left[x^2e^x - 2\left(xe^x - \int e^x dx\right)\right] \\ &= \frac{3}{2}x^3e^x - \frac{9}{2}\left[x^2e^x - 2\left(xe^x - e^x\right)\right] + c, \end{aligned}$$

from where

$$k(x) = \frac{3}{2}e^x\left[x^3 - 3x^2 + 6(x-1)\right] + c, c \in \mathbb{R}$$

Substituting in the homogeneous solution z_{Hom}, we obtain the general solution

$$z_{gle}(x) = \frac{1}{x^3}\left[\frac{3}{2}e^x\left[x^3 - 3x^2 + 6(x-1)\right] + c\right], c \in \mathbb{R}$$

and since $y_{gle} = z^{\frac{2}{3}}$ then the general solution is given by

$$y_{gle}(x) = \left(\frac{1}{x^3}\left[\frac{3}{2}e^x\left[x^3 - 3x^2 + 6(x-1)\right] + c\right]\right)^{\frac{2}{3}}, c \in \mathbb{R}$$

Example 11.8. *Solve*

$$y' - \frac{1}{2x}y = 5x^2y^5$$

Solution It is a Bernoulli's equation with $\alpha = 5$. We assume that $y(x) \neq 0$ and we divide by y^5

$$\frac{y'}{y^5} - \frac{1}{2xy^4} = 5x^2$$

$CV : z(x) = y^{-4}(x)$, so

$$\begin{aligned} z'(x) &= -4y'(x)y^{-5}(x) \\ &\Rightarrow \frac{y'(x)}{y^5(x)} = -\frac{1}{4}z'(x), \end{aligned}$$

then we replace in equation, we get

$$-\frac{1}{4}z' - \frac{1}{2x}z = 5x^2$$

which is a linear differential equation of order 1. Homogeneous equation

$$-\frac{1}{4}z' - \frac{1}{2x}z = 0$$
$$\Leftrightarrow \frac{z'}{z} = -\frac{2}{x} \Leftrightarrow \frac{dz}{z} = -\frac{2}{x}dx,$$

by integrating we get

$$\ln|z| = -2\ln|x| + c_1, c_1 \in \mathbb{R},$$
$$\Leftrightarrow |z| = e^{-2\ln|x|+c_1} = e^{c_1}e^{-2\ln|x|} = \frac{e^{c_1}}{x^2}$$

whence the homogeneous solution is given by

$$z_{\text{hom}}(x) = \frac{k}{x^2}, \textit{where } k = \pm e^{c_1} \in \mathbb{R}.$$

Variation of the constant

We vary the constant k. Let $z(x) = \frac{k(x)}{x^2}$ then $z' = \frac{k'}{x^2} - 2\frac{k}{x^3}$, and we substitute in the equation, which gives

$$-\frac{1}{4}\left(\frac{k'}{x^2} - 2\frac{k}{x^3}\right) - \frac{k}{2x^3} = 5x^2 \Leftrightarrow -\frac{k'}{4x^2} = 5x^2 \Leftrightarrow dk = -20x^4dx$$
$$\Rightarrow k = -4x^5 + c, c \in \mathbb{R}$$

and by replacing in the homogeneous solution z_{Hom}, we obtain the general solution of the equation

$$z_{gle}(x) = \frac{1}{x^2}\left[-4x^5 + c\right], c \in \mathbb{R}$$

and since $y_{gle} = z^{-\frac{1}{4}}$ then the general solution of first differential equation is given by

$$y_{gle}(x) = \left(\frac{1}{x^2}\left[-4x^5 + c\right]\right)^{-\frac{1}{4}} = \left(-4x^3 + cx^{-2}\right)^{-\frac{1}{4}}, c \in \mathbb{R}$$

Example 11.9. *Integrate the following differential equation*

$$y' + y - y^2(\cos x - \sin x) = 0$$

Solution

It is a Bernoulli's equation with $\alpha = 2$. We assume that $y(x) \neq 0$ and we divide by y^2

$$\frac{y'}{y^2} + \frac{1}{y} = \cos x - \sin x$$

$CV: z(x) = y^{-1}(x) = \frac{1}{y(x)}()$ hence

$$z'(x) = -y'(x)y^{-2}(x) = -\frac{y'(x)}{y^2(x)} \Rightarrow \frac{y'(x)}{y^2(x)} = -z'(x)$$

then we replace in the equation, so we get:

$$-z' + z = \cos x - \sin x$$

which is a linear differential equation of order 1.

Homogeneous equation

$$-z' + z = 0$$
$$\Leftrightarrow \frac{z'}{z} = 1 \Leftrightarrow \frac{dz}{z} = dx$$

by integrating we get

$$\ln|z| = x + c_1, c_1 \in \mathbb{R}$$
$$|z| = e^{x+c_1} = e^{c_1}e^x$$

from where the homogeneous solution is given by:

$$z_{\text{hom}}(x) = ke^x, \text{ where } k = \pm e^{c_1}$$

We vary the constant k*, then* $z'(x) = k'(x)e^x + k(x)e^x$*, and we replace in the corresponded equation, which gives*

$-k'e^x - ke^x + ke^x = \cos x - \sin x \Leftrightarrow -k'e^x = \cos x - \sin x \Leftrightarrow dk = e^{-x}(\sin x - \cos x)dx$

from where

$$k(x) = \int e^{-x}(\sin x - \cos x)dx = \int e^{-x}\sin x dx - \int e^{-x}\cos x dx$$

We use integration by parts

$$IPP : \begin{cases} u = \sin x \\ dv = e^{-x}dx \end{cases} \Rightarrow \begin{cases} du = \cos x dx \\ v = -e^{-x} \end{cases}$$
$$\Rightarrow \int e^{-x}\sin x dx = -e^{-x}\sin x + \int e^{-x}\cos x dx$$

thus

$$k(x) = -e^{-x}\sin x + \int e^{-x}\cos x dx - \int e^{-x}\cos x dx = -e^{-x}\sin x + c, c \in \mathbb{R}$$

and by replacing in the homogeneous solution z_{Hom}*, we obtain the general solution, of* *the Linear equation*

$$z_{gle}(x) = e^{x}\left(-e^{-x}\sin x + c\right) = ce^{x} - \sin x, c \in \mathbb{R}$$

and since $y_{gle} = z^{-1}$ *then the general solution of Bernoulli's equation is given by*

$$y_{gle}(x) = (ce^{x} - \sin x)^{-1} = \frac{1}{ce^{x} - \sin x}, c \in \ mathbbR$$

with $ce^{x} - \sin x \neq 0$

11.4 Riccati differential equations

Definition 11.10. *A Riccati equation is a nonlinear differential equation of order 1 of the form*

$$y'(x) = a(x)y^{2}(x) + b(x)y(x) + c(x)$$

where a, b *and* c *are given functions, continuous on an interval* I *of* $\mathbb{R}$, *with* $a(x) \neq 0, b(x) \neq 0$ *and* $c(x) \neq 0, \forall x \in I$.

Resolution method

Firstly, we need to find a particular solution of the Riccati equation noted $s(x)$ then make the following change of variables

$$y(x) = z(x) + s(x) \Rightarrow y'(x) = z'(x) + s'(x),$$

then we substitute in the equation, hence

$$\begin{aligned} z'(x) + s'(x) &= a(x)(z(x) + s(x))^{2} + b(x)(z(x) + s(x)) + c(x) \\ &= a(x)z^{2}(x) + b(x)z(x) + a(x)s^{2}(x) + b(x)s(x) + c(x) + 2a(x)z(x)s(x) \end{aligned}$$

and since $s(x)$ is a particular solution of the Riccati equation then it satisfies

$$s'(x) = a(x)s^{2}(x) + b(x)s(x) + c(x)$$

so

$$\begin{aligned} &\Leftrightarrow z'(x) = a(x)z^{2}(x) + b(x)z(x) + 2a(x)z(x)s(x) \\ &= a(x)z^{2}(x) + z(x)[b(x) + 2a(x)s(x)] \\ &\Leftrightarrow z'(x) - z(x)(b(x) + 2a(x)s(x)) - a(x)z^{2}(x) = 0 \end{aligned}$$

which is a Bernoulli equation, with $\alpha = 2$.

Example 11.10. *Solve*

$$xy' - y^2 + (2x+1)y - x^2 - 2x = 0, \text{ for } x \neq 0$$

Solution
It is a Riccati differential equation

$$y' = \frac{1}{x}y^2 - \left(2 + \frac{1}{x}\right)y + x + 2.$$

We notice that equation admits the function $s(x) = x$ as a particular solution and we make the following change of variables:

$$y = z + x \Rightarrow y' = z' + 1$$

then we replace in Riccati equation , hence

$$z' + 1 = \frac{1}{x}(z+x)^2 - \left(2 + \frac{1}{x}\right)(z+x) + x + 2$$

which is equivalent to

$$z' - \frac{1}{x}z^2 + \frac{z}{x} = 0 \Leftrightarrow xz' + z - z^2 = 0$$

which is a Bernoulli equation with $\alpha = 2$.
We divide it by z^2, hence

$$x\frac{z'}{z^2} + \frac{1}{z} - 1 = 0$$
$$CV : t = z^{-1} \Rightarrow t' = -\frac{z'}{z^2},$$

Thus

$$\Rightarrow -xt' + t = 1$$

which is a linear equation of order 1 with second member.
Homogeneous equation

$$-xt' + t = 0 \Leftrightarrow \frac{t'}{t} = \frac{1}{x} \Leftrightarrow \frac{dt}{t} = \frac{dx}{x}$$

where by integrating we get

$$\ln|t| = \ln|x| + c_1, c_1 \in \mathbb{R},$$
$$|t| = e^{\ln|x| + c_1} = e^{c_1}e^{\ln|x|},$$

Therefore

$$t_{hom}(x) = kx, \text{ with } k = \pm e^{c_1}.$$

Variation of the constant

We vary the constant k then $t' = k'x + k$, hence

$$\Rightarrow k' = -\frac{1}{x^2} \Leftrightarrow dk = -\frac{dx}{x^2}$$

then by integrating we get

$$k(x) = \frac{1}{x} + c, c \in \mathbb{R}$$

and replacing in t_{hom}, we get

$$t_{gle}(x) = \left(\frac{1}{x} + c\right) x = 1 + cx, \text{ with } c \in \mathbb{R}$$

therefore we have

$$z_{gle}(x) = \frac{1}{1 + cx}, \text{ with } c \in \mathbb{R}$$

and finally

$$y_{gle}(x) = z_{gle} + x = \frac{1}{1 + cx} + x, \text{ with } c \in \mathbb{R}.$$

Example 11.11. *Solve*

$$y' + \frac{y}{x} - y^2 = -\frac{1}{x^2}$$

such that $s(x) = \frac{1}{x}$ is a particular solution.

The solution is (check it)

$$y_{gle}(x) = \frac{x^2 + 2c}{x(-x^2 + 2c)}, \text{ with } c \ in \mathbb{R}$$

Exercise 11.1. *Integrate the following differential equation,*

$$y' - 2xy + y^2 = 2 - x^2$$

such that the function $s(x) = x + 1$ is a particular solution

Sorry, I can not tell you that the solution is

$$y_{gle}(x) = \frac{1 - x + 2ce^{2x}(x + 1)}{2ce^{2x} - 1}, \text{ with } c \in \mathbb{R}.$$

CHAPTER

12

LINEAR DIFFERENTIAL EQUATIONS OF ORDER 2 WITH CONSTANT COEFFICIENTS

12.1 Linear differential equations of order 2 with constant coefficients

Definition 12.1. *A linear differential equation of order 2 with constant coefficients is any equation of the form*

$$(E) : ay'' + by' + cy = f(x)$$

where $a, b, c \in \mathbb{R}, a \neq 0$ and f is a continuous function on an interval I of $\mathbb{R}$.

Definition 12.2. *We call the homogeneous equation or equation without the second member the equation:*

$$(F) : ay'' + by' + cy = 0$$

Theorem 12.1. *The general solution y yle of the nonhomogeneous differential equation is the sum of a particular solution y_p of this nonhomogeneous equation and the solution y_{Hom} of the homogeneous equation $y_{gle} = y_{Hom} + y_p$*

Proof. We verify that $y_{\text{hom}} + y_p$ is a solution of equation (E) , in fact

$$\begin{aligned}
&a\left(y_{\text{hom}}+y_p\right)''+b\left(y_{\text{hom}}+y_p\right)'+c\left(y_{\text{hom}}+y_p\right)\\
&=\left(ay''_{\text{hom}}+by'_{\text{hom}}+cy_{\text{hom}}\right)+\left(ay''_p+by'_p+cy_p\right)\\
&=f(x)
\end{aligned}$$

Conversely, if y_p is a particular solution of equation (E) and y is another solution of equation (F), then their difference is a solution of the homogeneous equation, indeed

$$\begin{aligned}
&a\left(y-y_p\right)''+b\left(y-y_p\right)'+c\left(y-y_p\right)\\
&=\left(ay''+by'+cy\right)-\left(ay''_p+by'_p+cy_p\right)\\
&=f(x)-f(x)=0
\end{aligned}$$

□

Remark 12.1.
- *The zero solution $y = 0$ is a trivial solution of the homogeneous equation.*
- *If y_1 and y_2 are two solutions of the homogeneous equation then for all α, β in $\mathbb{R}$, than $\alpha y_1 + \beta y_2$ is a solution of the homogeneous equation too.*

12.2 Method of resolution

Step 1

We first solve the homogeneous equation (without right-hand side):

$$ay'' + by' + cy = 0, \quad (6)$$

We set $y = e^{rx}$, where r is a constant, hence $y' = re^{rx}$ and $y'' = r^2e^{rx}$, then we substitute in (6) hence

$$e^{rx}\left(ar^2 + br + c\right) = 0$$

which is equivalent to

$$ar^2 + br + c = 0 \quad (6.1)$$

Equation (6.1) is called the "characteristic equation" of the differential equation (6), we solve this equation by first calculating its discriminant $\Delta = b^2 - 4ac$, where we distinguish 3 cases namely Let $\Delta = b^2 - 4ac$ be the discriminant of the characteristic equation. The differential equation $(E) : ay'' + by' + cy = 0$.
Case 1: If $\Delta > 0$ then equation (6.1) admits two distinct real solutions $r_1 = \frac{-b-\sqrt{\Delta}}{2a}$ and

$r_2 = \frac{-b+\sqrt{\Delta}}{2a}$ and in this case the homogeneous solution of equation (E) is in the form

$$y_{\text{hom}}(x) = Ae^{r_1 x} + Be^{r_2 x}, A, B \in \mathbb{R}.$$

Case 2: If $\Delta = 0$ then equation (6.1) admits a double solution : $r = \frac{-b}{2a}$ and in this case the homogeneous solution of equation (E) is in the form

$$y_{\text{hom}}(x) = e^{rx}(A + Bx), A, B \in \mathbb{R}.$$

Case 3: If $\Delta < 0$ then equation (6.1) admits two complex conjugate solutions $r_1 = \beta + i\omega$ and $r_2 = \beta - i\omega$ and in this case the homogeneous solution of equation (E) is in the form

$$y_{\text{hom}}(x) = e^{\beta x}(C \cos \omega x + D \sin \omega x), C, D \in \mathbb{R}.$$

Justification:

- If $\Delta \geq 0$ then we assume that r is a real solution of the characteristic equation (6.1) and we make the change of variable $y(x) = ae^{rx}z(x)$. We differentiate twice and substitute in (E), then we get $e^{rx}\left[(2ar + b)z' + z''\right] = 0$, which is equivalent to

$$(2ar + b)z' + az'' = 0$$

- If r is a double solution of the characteristic equation then

$$r = \frac{-b}{2a} \Leftrightarrow 2ar + b = 0$$

and in this case

$$z'' = 0$$

hence by integrating twice we have

$$\begin{aligned} &z(x) = c_1 x + c_2, \text{ with } c_1, c_2 \in \mathbb{R} \\ &\Rightarrow y(x) = ae^{rx}\left(c_1 x + c_2\right) = e^{rx}\left(ac_1 x + ac_2\right) \end{aligned}$$

hence by setting $ac_1 = A$ and $ac_2 = By(x) = e^{rx}(Ax + B)$, with $A, B \in \mathbb{R}$.

- If r is a simple solution of the characteristic equation (6.1) then $2ar+ b \neq 0$ and the other solution would be $r' = -\left(r + \frac{b}{a}\right)$ because their sum is equal to $-\frac{b}{a}$ and in this case

$$\frac{z''}{z'} = -\frac{(2ar + b)}{a} = -2r - \frac{b}{a}$$

hence by integrating twice we have

$$\begin{aligned}
&\ln|z'| = -\left(2r + \frac{b}{a}\right)x + c_1, \text{ with } c_1 \in \mathbb{R} \\
&\Rightarrow z' = k_1 e^{-\left(2r+\frac{b}{a}\right)x}, \text{ with } k_1 = \pm e^{c_1} \\
&\Rightarrow z(x) = \frac{-k_1}{\left(2r + \frac{b}{a}\right)} e^{-\left(2r+\frac{b}{a}\right)x} + k_2, \text{ with } k_2 \in \mathbb{R}
\end{aligned}$$

Thus

$$y(x) = \frac{-ak_1}{\left(2r + \frac{b}{a}\right)} e^{-\left(r+\frac{b}{a}\right)x} + k_2 e^{rx}$$

then by setting $\frac{-ak_1}{\left(2r+\frac{b}{a}\right)} = A$ and $k_2 = B$, we have

$$y(x) = Ae^{r'x} + Be^{rx}, A, B \in \mathbb{R}$$

- If $\Delta < 0$ then equation (6.1) admits two complex conjugate solutions $r_1 = \beta + i\omega$ and $r_2 = \beta - i\omega$ and in this case equation (6) admits two solutions

$$\begin{aligned}
y_1 &= e^{(\beta+i\omega)x} = e^{\beta x}(\cos\omega x + i\sin\omega x) \\
y_2 &= e^{(\beta-i\omega)x} = e^{\beta x}(\cos\omega x - i\sin\omega x)
\end{aligned}$$

what yields to

$$\begin{aligned}
Y_1 &= \frac{1}{2}(y_1 + y_2) = e^{\beta x}\cos\omega x \\
Y_2 &= \frac{1}{2i}(y_1 - y_2) = e^{\beta x}\sin\omega x
\end{aligned}$$

and since any linear combination is also a solution of the homogeneous equation (6), then

$$\begin{aligned}
y(x) &= Ce^{\beta x}\cos\omega x + De^{\beta x}\sin\omega x \\
&= e^{\beta x}(C\cos\omega x + D\sin\omega x)
\end{aligned}$$

with $C, D \in \mathbb{R}$

12.3 Form of particular solution for second-order D.E. with constant coefficients

Consider the second-order linear ODE with constant coefficients:

$$a_2 y'' + a_1 y' + a_0 y = g(x),$$

where a_2, a_1, a_0 are constants, and $g(x)$ is the non-homogeneous term. Recall that the associated homogeneous equation is:

$$a_2y'' + a_1y' + a_0y = 0.$$

The characteristic equation is:

$$a_2r^2 + a_1r + a_0 = 0.$$

Solve for the roots r_1 and r_2. Depending on the nature of the roots:

- Real and distinct roots: $y_h(x) = C_1e^{r_1x} + C_2e^{r_2x}$,
- Repeated real root: $y_h(x) = C_1e^{r_1x} + C_2xe^{r_1x}$,
- Complex conjugate roots: $y_h(x) = e^{\alpha x}\big(C_1\cos(\beta x) + C_2\sin(\beta x)\big)$.

Form of the Particular Solution

The form of the particular solution $y_p(x)$ depends on $g(x)$. Use the method of undetermined coefficients to guess $y_p(x)$. The common cases are:

Case 1: $g(x) = P_m(x)$ (Polynomial of degree m)

If $g(x) = P_m(x) = a_mx^m + a_{m-1}x^{m-1} + \cdots + a_0$, assume:

$$y_p(x) = A_mx^m + A_{m-1}x^{m-1} + \cdots + A_1x + A_0.$$

If $a_0 = 0$, multiply $y_p(x)$ by x to avoid overlap with the homogeneous solution.

Case 2: $g(x) = e^{kx}$ (Exponential function)

If $g(x) = e^{kx}$, assume:

$$y_p(x) = Ae^{kx}.$$

If e^{kx} is already a solution to the homogeneous equation, multiply by x^n, where n is the smallest integer such that x^ne^{kx} is not a solution of the homogeneous equation.

Case 3: $g(x) = \sin(kx)$ or $\cos(kx)$

If $g(x) = \sin(kx)$ or $\cos(kx)$, assume:

$$y_p(x) = A\cos(kx) + B\sin(kx).$$

If $\sin(kx)$ or $\cos(kx)$ is already a solution to the homogeneous equation, multiply by x.

Case 4: $g(x) = P_m(x)e^{kx}$ **(Product of polynomial and exponential)**

If $g(x) = P_m(x)e^{kx}$, assume:

$$y_p(x) = \big(A_m x^m + A_{m-1}x^{m-1} + \cdots + A_1 x + A_0\big)e^{kx}.$$

If e^{kx} is already a solution to the homogeneous equation, multiply by x^n, where n is the smallest integer such that $x^n e^{kx}$ is not a solution of the homogeneous equation.

Case 5: $g(x) = P_m(x)\sin(kx)$ **or** $P_m(x)\cos(kx)$

If $g(x) = P_m(x)\sin(kx)$ or $P_m(x)\cos(kx)$, assume:

$$y_p(x) = \big(A_m x^m + A_{m-1}x^{m-1} + \cdots + A_1 x + A_0\big)\cos(kx) + \big(B_m x^m + B_{m-1}x^{m-1} + \cdots + B_1 x + B_0\big)\sin(kx).$$

If $\sin(kx)$ or $\cos(kx)$ is already a solution to the homogeneous equation, multiply by x.

Step 3: Combine Solutions

The general solution to the ODE is the sum of the complementary solution $y_h(x)$ and the particular solution $y_p(x)$:

$$y(x) = y_h(x) + y_p(x).$$

Examples

- If $g(x) = 3e^{2x}$, assume $y_p(x) = Ae^{2x}$. If e^{2x} is a solution to the homogeneous equation, use $y_p(x) = Axe^{2x}$.
- If $g(x) = 5x^2 + 3x + 1$, assume $y_p(x) = Ax^2 + Bx + C$.
- If $g(x) = \sin(3x)$, assume $y_p(x) = A\cos(3x) + B\sin(3x)$.
- If $g(x) = xe^{-x}$, assume $y_p(x) = (Ax + B)e^{-x}$.
- $2y'' - y' = 3x^2 + 2x - 1$,
- $y'' + y' + y = 5x + 1$,
- $y'' - 6y' + 9y = -2e^{3x}$.

Solutions of some examples

1.

$$2y'' - y' = 3x^2 + 2x - 1$$

Homogeneous equation

$$2y'' - y' = 0$$

Characteristic equation

$$2r^2 - r = 0$$

We have $\Delta = 1 > 0$ or we can simply note directly that

$$r(2r - 1) = 0$$

then equation admits two distinct real solutions: $r_1 = 0$ and $r_2 = \frac{1}{2}$ hence the homogeneous solution

$$y_{\text{hom}}(x) = A + Be^{\frac{1}{2}x}, \quad A, B \in \mathbb{R}$$

Particular solution y_p

To calculate the particular solution of equation we note that the second member $f(x)$ is a polynomial of degree 2, and that 0 is a solution of the characteristic equation so the particular solution y_p of equation will have the following form

$$y_p(x) = x\left(ax^2 + bx + c\right) = ax^3 + bx^2 + cx$$

where a, b, c are real constants to be determined, then we differentiate y_p twice and replace in equation.

$$y_p' = 3ax^2 + 2bx + c, y_p'' = 6ax + 2b$$
$$\Rightarrow -3ax^2 + x(12a - 2b) + 4b - c = 3x^2 + 2x - 1$$

by performing an identification between the two members of the equation, we obtain the following system

$$\begin{cases} -3a = 3 \\ 12a - 2b = 2 \\ 4b - c = -1 \end{cases}$$

SO

$$\begin{cases} a = -1 \\ b = -7 \\ c = -27 \end{cases}$$

therefore the particular solution of equation is given by

$$y_p(x) = x\left(-x^2 - 7x - 27\right)$$

and therefore the general solution is

$$y_{gle}(x) = \left(A + Be^{\frac{1}{2}x}\right) - x\left(x^2 + 7x + 27\right), A, B \in \mathbb{R}$$

2.

$$y'' + y' + y = 5x + 1$$

We have $\Delta = -3 < 0 \Rightarrow \Delta = (\sqrt{3}i)^2$ then equation admits two complex conjugate solutions $r_1 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$ and $r_2 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$ from which the homogeneous solution is

$$y_{\text{hom}}(x) = e^{-\frac{1}{2}x}\left[A\cos\left(\frac{\sqrt{3}}{2}x\right) + B\sin\left(\frac{\sqrt{3}}{2}x\right)\right], \quad A, B \in \mathbb{R}.$$

Particular solution y_p

To calculate the particular solution of equation, we note the form of the second member, we have $f(x) = 5x + 1$, a polynomial of degree 1 and we note that $r = 0$ is not a solution of the characteristic equation then the particular solution will have the following form

$$y_p(x) = ax + b$$

where a and b are real constants to be determined, then we differentiate twice y_p and we replace in equation,

$$y_p' = a, y_p'' = 0$$
$$\Rightarrow ax + (a + b) = 5x + 1$$

then by identification we obtain the system $\begin{cases} a = 5 \\ a + b = 1 \end{cases}$

SO

$$\begin{cases} a = 5 \\ b = -4 \end{cases}$$

therefore the particular solution of the equation is

$$y_p(x) = 5x - 4$$

and hence the general solution is

$$y_{\text{gle}}(x) = e^{-\frac{1}{2}x}\left[A\cos\left(\frac{\sqrt{3}}{2}x\right) + B\sin\left(\frac{\sqrt{3}}{2}x\right)\right] + 5x - 4, \quad A, B \in \mathbb{R}.$$

3.

$$y'' - 6y' + 9y = -2e^{3x}$$

Homogeneous equation

$$y'' - 6y' + 9y = 0$$

Characteristic equation

$$r^2 - 6r + 9 = 0$$

We have $\Delta = 0$ or it suffices to note directly that

$$(r-3)^2 = 0$$

then equation admits a real double solution $r_0 = 3$, from which the homogeneous solution is

$$y_{\text{hom}}(x) = Ae^{3x} + Bxe^{3x} = e^{3x}(A + Bx), \quad A, B \in \mathbb{R}$$

Particular solution y_p To calculate the particular solution of the equation, we note the form of the second member $f(x) = -2e^{3x}$, and as $r = 3$ is a double solution, therefore of multiplicity 2, of the characteristic the equation, then the particular solution will have the following form

$$y_p(x) = \alpha x^2 e^{3x}$$

where α is a real constant (or polynomial of degree 0) that must be determined. We derive y_p twice and replace in the equation,

$$y_p' = \alpha e^{3x}\left(3x^2 + 2x\right), y_p'' = \alpha e^{3x}\left(9x^2 + 12x + 2\right)$$
$$3.53) \Rightarrow 2\alpha e^{3x} = -2e^{3x}$$

then by identification we have

$$\alpha = -1$$

therefore the particular solution of the equation is

$$y_p(x) = -x^2 e^{3x}$$

and therefore the general solution is

$$y_{gle}(x) = e^{3x}(A + Bx) - x^2 e^{3x}, \quad A, B \in \mathbb{R}$$

Exercise 12.1. *Solve the following differential equations*

1. $y'' + 2y' + 5y = \sin(2x)$.
2. $y'' - 5y' + 6y = e^x(x \sin x + \cos x)$.

Solution

1.

$$y'' + 2y' + 5y = \sin(2x)$$

Homogeneous equation

$$y'' + 2y' + 5y = 0$$

Characteristic equation

$$r^2 + 2r + 5 = 0$$

We have $\Delta = -16 < 0 \Rightarrow \Delta = (4i)^2$ then the equation admits two complex conjugate solutions

$$r_1 = -1 - 2i \text{ and } r_2 = -1 + 2i$$

hence the homogeneous solution is

$$y_{\text{hom}}(x) = e^{-x}[A\cos(2x) + B\sin(2x)], \quad A, B \in \mathbb{R}.$$

Particular solution y_p To calculate the particular solution of the equqtion, we note the second member $f(x) = \sin(2x)$, and that $r = 2i$ is not a solution of the characteristic equation then the particular solution takes the following form:

$$y_p(x) = a\cos(2x) + b\sin(2x)$$

where a and b are real constants to be determined. We derive twice y_p

$$y_p' = 2(b\cos(2x) - a\sin(2x)), y_p'' = -4(a\cos(2x) + b\sin(2x))$$

and the equation becomes

$$(a + 4b)\cos(2x) + (-4a + b)\sin(2x) = \sin(2x)$$
$$\Leftrightarrow (a + 4b)\cos(2x) + (-4a + b - 1)\sin(2x) = 0$$

thus, we deduce

$$\begin{cases} a + 4b = 0 \\ -4a + b - 1 = 0 \end{cases}$$

so

$$\begin{cases} a = -\frac{4}{17} \\ b = \frac{1}{17} \end{cases}$$

therefore the particular solution of the equation is

$$y_p(x) = \frac{1}{17}(\sin(2x) - 4\cos(2x))$$

and therefore the general solution is
$y_{gle}(x) = e^{-x}[A\cos(2x) + B\sin(2x)] + \frac{1}{17}(\sin(2x) - 4\cos(2x)), \quad A, B \in \mathbb{R}$.
2.

$$y'' - 5y' + 6y = e^x(x\sin x + \cos x)$$

Homogeneous equation

$$y'' - 5y' + 6y = 0$$

Characteristic equation

$$r^2 - 5r + 6 = 0$$

We have $\Delta = 1 > 0$ then the equation admits two distinct real solutions $r_1 = 2$ and $r_2 = 3$ hence the homogeneous solution

$$y_{\text{hom}}(x) = Ae^{2x} + Be^{3x}, \quad A, B \in \mathbb{R}$$

Particular solution y_p

To calculate the particular solution of the equation, we have the form of the second member, $f(x) = e^x(x\sin x + \cos x)$, and we note that $r = 1 + i$ is not a solution of the characteristic equation, so the particular solution will have the following form

$$y_p(x) = e^x[(ax + b)\sin x + (cx + d)\cos x]$$

where a, b, c and d are real constants to be determined, then we calculate the derivative of y_p twice

$$\begin{aligned} y_p' &= e^x[\sin x((a - c)x + a + b - d) + \cos x((a + c)x + b + c + d)], \\ y_p'' &= e^x[\sin x(-2cx + 2a + -2c - 2d) + \cos x(2ax + 2a + 2b + 2c)] \end{aligned}$$

and we replace in the equation, hence

$$\begin{aligned} [x(a + 3c) - 3a + b - 2c + 3d]\sin x + [x(-3a + c) + 2a - 3b - 3c + d]\cos x \\ = x\sin x + \cos x \end{aligned}$$

and we obtain the following system

$$\begin{cases} a + 3c = 1 \\ -3a + c = 0 \\ -3a + b - 2c + 3d = 0 \\ 2a - 3b - 3c + d = 1 \end{cases}$$

and after solving the system we find

$$\begin{cases} a = \frac{1}{10}, \quad b = -\frac{21}{50} \\ c = \frac{3}{10}, \quad d = \frac{11}{25} \end{cases}$$

therefore the particular solution of the equation is

$$y_p(x) = e^x\left[\left(\frac{1}{10}x - \frac{21}{50}\right)\sin x + \left(\frac{3}{10}x + \frac{11}{25}\right)\cos x\right]$$

and so the general solution is

$$y_{gle}(x) = Ae^{2x} + Be^{3x} + \frac{1}{10}e^x.$$

Note: In general, one can look for the particular solution using the method of variation of constants, especially if the coefficients of the differential equation are not constant or if the second member $f(x)$ is different from the forms given above.

Indeed, by writing the homogeneous solution

$$y_{\text{hom}} = Ay_1 + By_2$$

where y_1 and y_2 are two linearly independent solutions of the homogeneous equation, we seek a general solution of the nonhomogeneous equqtion in the form

$$y_{gle} = Ay_1 + By_2$$

considering A and B as two functions that satisfy

$$A'y_1 + B'y_2 = 0$$

then by differentiating y_p twice and replacing it in 3.46, we obtain

$$a\left(A'y_1' + B'y_2'\right) = f(x)$$

which gives us the system

$$\begin{cases} A'y_1 + B'y_2 = 0 \\ A'y_1' + B'y_2' = \frac{1}{a}f(x) \end{cases}$$

which we solve to have A' and B' then by integration A and B and finally the general solution y_{gle}.

Examples 12.1. 1. *Solve the following differential equation using the method of variation of constants.*

$$y'' + y = \frac{1}{\sin^3 x}$$

Homogeneous equation

$$y'' + y = 0$$

Characteristic equation

$$r^2 + 1 = 0 \Leftrightarrow (r+i)(r-i) = 0$$

then the equation admits two complex solutions $r_1 = i$ and $r_2 = -i$ hence the homogeneous solution

$$y_{hom}(x) = A\cos x + B\sin x, \quad A, B \in \mathbb{R}.$$

Variation of constants. We note that $y_1 = \cos x$ and $y_2 = \sin x$ are two linearly independent solutions of the homogeneous equation, so we seek a general solution of nonhomogeneous equation in the form

$$y = A\cos x + B\sin x$$

such that A and B are two functions that verify the system

$$\begin{cases} A'\cos x + B'\sin x = 0, \\ -A'\sin x + B'\cos x = \frac{1}{\sin^3 x}. \end{cases}$$

Multiplying the 1st equation by $\sin x$ and the 2nd by $\cos x$, then adding up, we have

$$B' = \frac{\cos x}{\sin^3 x} \Leftrightarrow dB = \frac{\cos x}{\sin^3 x}dx$$

we integrate on both sides, making the change of variables $t = \sin x \Rightarrow dt = \cos x dx$ then we get

$$B = -\frac{1}{2\sin^2 x} + c_1, c_1 \in \mathbb{R}$$

then replacing in the system, we have

$$A' = -\frac{1}{\sin^2 x} \Leftrightarrow dA = -\frac{1}{\sin^2 x}dx$$

and after integration we obtain

$$A = \cot x + c_2 = \frac{\cos x}{\sin x} + c_2, c_2 \in \mathbb{R}$$

therefore,

$$\begin{aligned} y_{gle} &= \frac{\cos^2 x}{\sin x} + c_2\cos x - \frac{1}{2\sin x} + c_1\sin x \\ &= \frac{2\cos^2 x - 1}{2\sin x} + c_1\sin x + c_2\cos x = \frac{\cos 2x}{2\sin x} + c_1\sin x + c_2\cos x \end{aligned}$$

with $c_1, c_2 \in \mathbb{R}$

2. *Solving the equation*

$$y'' - \frac{1}{x}y' = x$$

Homogeneous equation

$$y'' - \frac{1}{x}y' = 0 \Leftrightarrow \frac{y''}{y'} = \frac{1}{x}$$

which gives

$$\ln|y'| = \ln|x| + c_1, c_1 \in \mathbb{R}$$

hence

$$y' = c_2 x, \ \textit{with } c_2 = \pm e^{c_1} \in \mathbb{R}$$

we integrate to find

$$y_{\text{hom}} = Ax^2 + B, \ \textit{avec } A, B \in \mathbb{R} \textit{ et } A = \frac{c_2}{2}.$$

Variation of constants

This is the second method to provide solution of the general equation. Let be

$$y_1 = x^2 \text{ and } y_2 = 1$$

such that y_1 and y_2 are two linearly independent solutions of the homogeneous equation and we seek a general solution of the general equation in the form

$$y_p = Ay_1 + By_2$$

such that A and B are two functions that satisfy

$$\begin{cases} A'y_1 + B'y_2 = 0 \\ A'y_1' + B'y_2' = x \end{cases}$$

the resolution of this system gives

$$A' = \frac{1}{2}, \quad B' = -\frac{x^2}{2},$$

which gives after integration

$$A = \frac{x}{2} + k_1, B = -\frac{x^3}{6} + k_2, \text{ with } k_1, k_2 \in \mathbb{R}$$

From where

$$\begin{aligned} y_{\text{gle}} &= \left(\frac{x}{2} + k_1\right) x^2 - \frac{x^3}{6} + k_2 \\ &= k_2 + k_1 x^2 + \frac{x^3}{3} \text{ with } k_1, k_2 \in \mathbb{R}. \end{aligned}$$

Note To determine the constants k_1 and k_2, it is sufficient to give two initial conditions, $y_1 = y\,(x_0)$ and $y_2 = y'\,(x_0)$.

Superposition principle

Theorem 12.2. *Given the linear differential equation of order 2*

$$ay'' + by' + cy = f_1(x) + f_2(x)$$

where $a, b, c \in \mathbb{R}$ and f_1, f_2 are two continuous functions on an interval I of $\mathbb{R}$. The particular solution y_p of (3.63) can be expressed by the sum of the two particular solutions y_{p_1} and y_{p_2} of the respective differential equations:

$$ay'' + by' + cy = f_1(x)$$

and

$$ay'' + by' + cy = f_2(x)$$

such that

$$y_p = y_{p_1} + y_{p_2}$$

Proof. We can easily verify that $y_{p_1} + y_{p_2}$ is a solution of the nonhomogeneous differential equation, indeed because of the linearity of the equation, we have

$$\begin{aligned} a\,(y_{p_1} + y_{p_2})'' + b\,(y_{p_1} + y_{p_2})' + c\,(y_{p_1} + y_{p_2}) &= \left(ay''_{p_1} + by'_{p_1} + cy_{p_1}\right) + \left(ay''_{p_2} + by'_{p_2} + cy_{p_2}\right) \\ &= f_1(x) + f_2(x). \end{aligned}$$

Note For the general solution y_{gle} of the nonhomogeneous differential equation, it is sufficient to write the homogeneous equation and then solve it to obtain the homogeneous solution y_{hom} and we obtain

$$y_{gle} = y_{\text{hom}} + y_{p_1} + y_{p_2}.$$

□

Example 12.1. *Solve the following differential equations*

1. $y'' - 3y' = (x+2)e^{2x} + (3\sin x + 2\cos x)$.

2. $y'' + 2y' + 2y = 2x - \sin x$. *3.* $y'' - 4y' + 3y = 3x + 2 + 4e^{x} + 5e^{-x}$.

Solution

1. One has

$$y'' - 3y' = (x+2)e^{2x} + 3\sin x + 2\cos x$$

Homogeneous equation:

$$y'' - 3y' = 0$$

Characteristic equation:

$$r^2 - 3r = 0$$

We have $\Delta = 9 > 0$ then the equation admits two distinct real solutions $r_1 = 0$ and $r_2 = 3$ hence the homogeneous solution

$$y_{\text{hom}}(x) = A + Be^{3x}, \quad A, B \in \mathbb{R}$$

Particular solution y_p

To calculate the particular solution y_p of nonhomogeneous equation, we note the second member $f(x) = f_1(x) + f_2(x)$, or $f_1(x) = (x+2)e^{2x}$ and $f_2(x) = 3\sin x + 2\cos x$, then we will use the superposition principle such that

$$y_p = y_{p_1} + y_{p_2}.$$

where y_{p_1} and y_{p_2} are the particular solutions of the respective differential equations

$$y'' - 3y' = (x+2)e^{2x}$$

and

$$y'' - 3y' = 3\sin x + 2\cos x$$

Calculation of y_{p_1}. We have

$$y'' - 3y' = (x+2)e^{2x}$$

We note that $r = 2$ is not not a solution to the characteristic equation then the particular solution to the equation, will have the following form

$$y_{p_1}(x) = e^{2x}(ax+b)$$

where a, b are real constants to be determined. We derive twice y_{p_1}

$$\begin{aligned} y'_{p_1} &= e^{2x}[2ax + a + 2b], \\ y''_{p_1} &= e^{2x}[4ax + 4a + 4b], \end{aligned}$$

and we replace in the equation, hence

$$\begin{aligned} & e^{2x}[-2ax + (a-2b)] = e^{2x}(x+2) \\ \Leftrightarrow\ & e^{2x}[x(-2a-1) + (a-2b-2)] = 0 \end{aligned}$$

then we obtain the system next

$$\begin{cases} -2a - 1 = 0 \\ a - 2b - 2 = 0 \end{cases}$$

which gives

$$\begin{cases} a = -\frac{1}{2} \\ b = -\frac{5}{4} \end{cases}$$

therefore the particular solution of the nonhomogeneous equation is

$$y_{p_1}(x) = -\frac{1}{4}e^{2x}(2x+5)$$

Calculation of y_{p_2}. We have

$$y'' - 3y' = 3\sin x + 2\cos x$$

We note that $r = i$ is not a solution to the characteristic equation, so the particular solution to the equation will have the following form

$$y_{p_2}(x) = \alpha \cos x + \beta \sin x$$

where α, β are real constants to be determined. We derive twice y_{p_2}

$$\begin{aligned} y'_{p_2} &= -\alpha \sin x + \beta \cos x, \\ y''_{p_2} &= -\alpha \cos x - \beta \sin x, \end{aligned}$$

and we replace in the equation, from which

$$(3\alpha - \beta) \sin x + (-\alpha - 3\beta) \cos x = 3 \sin x + 2 \cos x$$

then by identification we obtain the following system

$$\begin{cases} 3\alpha - \beta = 3 \\ -\alpha - 3\beta = 2 \end{cases}$$

and after solving the system we find:

$$\begin{cases} \alpha = \frac{7}{10} \\ \beta = -\frac{9}{10} \end{cases}$$

therefore the particular solution of the equation is

$$y_{p_2}(x) = \frac{7}{10} \cos x - \frac{9}{10} \sin x$$

Hence the particular solution of the equation is

$$y_p(x) = -\frac{1}{4} e^{2x}(2x+5) + \left(\frac{7}{10} \cos x - \frac{9}{10} \sin x \right)$$

and hence the general solution of the equation is

$$y_{gle}(x) = A + Be^{3x} - \frac{1}{4} e^{2x}(2x+5) + \left(\frac{7}{10} \cos x - \frac{9}{10} \sin x \right), \quad A, B \in \mathbb{R}$$

2. One has

$$y'' + 2y' + 2y = 2x - \sin x$$

Homogeneous equation

$$y'' + 2y' + 2y = 0$$

Characteristic equation

$$r^2 + 2r + 2 = 0$$

We have $\Delta = -4 < 0$ then the equation admits two complex solutions

$$r_1 = -1 - i \text{ and } r_2 = -1 + i$$

hence the homogeneous solution is

$$y_{\text{hom}}(x) = e^{-x}(A\cos x + B\sin x), \quad A, B \in \mathbb{R}$$

Particular solution y_p

To calculate the particular solution y_p of the equation, we note the right-hand side $f(x) = f_1(x)+f_2(x)$, or $f_1(x) = 2x$ and $f_2(x) = -\sin x$, then we can use the superposition principle

$$y_p = y_{p_1} + y_{p_2}$$

where y_{p_1} and y_{p_2} are the particular solutions of the respective differential equations

$$y'' + 2y' + 2y = 2x$$

and

$$y'' + 2y' + 2y = -\sin x$$

Calculation of y_{p_1}. We have

$$y'' + 2y' + 2y = 2x$$

We note that $r = 0$ is not a solution to the characteristic the equation, so the particular solution to the equation will have the following form

$$y_{p_1}(x) = ax + b$$

where a, b are real constants to be determined. We derive twice y_{p_1}

$$y'_{p_1} = a$$
$$y''_{p_1} = 0$$

and we substitute in the equation. From where

$$ax + a + b = x$$

then we obtain the following system

$$\begin{cases} a = 1 \\ a + b = 0 \end{cases}$$

hence

$$a = 1 \text{ and } b = -1$$

therefore the particular solution of the equation is

$$y_{p_1}(x) = x - 1$$

Calculation of y_{p_2}. We have

$$y'' + 2y' + 2y = -\sin x$$

We note that $r = i$ is not a solution to the characteristic the equation so the particular solution to equation equation will have the following form.

$$y_{p_2}(x) = \alpha \cos x + \beta \sin x$$

where α, β are real constants to be determined. We derive twice y_{p_2}

$$\begin{aligned} y'_{p_2} &= -\alpha \sin x + \beta \cos x, \\ y''_{p_2} &= -\alpha \cos x - \beta \sin x, \end{aligned}$$

and we replace in the equation, from which

$$(\alpha + 2\beta) \cos x + (-2\alpha + \beta) \sin x = -\sin x$$

then by identification we obtain the following system

$$\begin{cases} \alpha + 2\beta = 0 \\ -2\alpha + \beta = -1 \end{cases}$$

hence

$$\alpha = \frac{2}{5} \text{ and } \beta = -\frac{1}{5}$$

Thus the particular solution of the equation is

$$y_{p_2}(x)=\frac{2}{5}\cos x-\frac{1}{5}\sin x$$

Hence the particular solution of the equation is

$$y_p(x)=(x-1)+\left(\frac{2}{5}\cos x-\frac{1}{5}\sin x\right)$$

and therefore the general solution of the equation is written

$$y_{gle}(x)=e^{-x}(A\cos x+B\sin x)+(x-1)+\left(\frac{2}{5}\cos x-\frac{1}{5}\sin x\right), A, B\in\mathbb{R}.$$

3. Let be

$$y''-4y'+3y=(3x+2)+4e^{x}+5e^{-x}$$

Homogeneous equation

$$y''-4y'+3y=0$$

Characteristic equation

$$r^2-4r+3=0$$

We have $\Delta=4>0$ then the equation admits two distinct real solutions $r_1=1$ and $r_2=3$ hence the homogeneous solution

$$y_{\text{hom}}(x)=Ae^{x}+Be^{3x},\quad A, B\in\mathbb{R}$$

Particular solution y_p

To calculate the particular solution y_p of the equation, we note the right-hand side, $f(x)=f_1(x)+f_2(x)+f_3(x)$, or $f_1(x)=3x+2, f_2(x)=4e^{x}$ and $f_3(x)=5e^{-x}$ then we will use the superposition principle, such that

$$y_p=y_{p_1}+y_{p_2}+y_{p_3}$$

where y_{p_1}, y_{p_2} and y_{p_3} are the particular solutions of the respective differential equations

$$\begin{aligned} y''-4y'+3y&=3x+2,\\ y''-4y'+3y&=4e^{x}, \end{aligned}$$

and

$$y'' - 4y' + 3y = 5e^{-x}$$

Calculation of y_{p_1}. We have

$$y'' - 4y' + 3y = 3x + 2$$

We note that $r = 0$ is not a solution to the characteristic equation so the particular solution to the equation will have the following form.

$$y_{p_1}(x) = ax + b$$

where a, b are real constants to be determined. We derive twice y_{p_1}

$$y'_{p_1} = a$$
$$y''_{p_1} = 0$$

and we replace in the equation, to find

$$3ax - 4a + 3b = 3x + 2$$

and the identification leads to the following system

$$\begin{cases} 3a = 3 \\ -4a + 3b = 2 \end{cases}$$

hence

$$a = 1 \text{ and } b = 2$$

therefore the particular solution of the equation is written

$$y_{p_1}(x) = x + 2.$$

Calculation of y_{p_2}. We have

$$y'' - 4y' + 3y = 4e^{x}$$

We note that $r = 1$ is a solution of the characteristic equation. So the particular solution of the equation is written as

$$y_{p_2}(x) = \alpha x e^{x}$$

where α is a real constant to be determined. We derive twice y_{p_2}

$$y'_{p_2} = \alpha(1+x)e^x,$$
$$y''_{p_2} = \alpha(2+x)e^x,$$

and we replace in the equation, hence

$$\alpha = -2$$

therefore the particular solution of the equation is written as

$$y_{p_2}(x) = -2xe^x$$

Calculation of y_{p_3}.. We have

$$y'' - 4y' + 3y = 5e^{-x}.$$

We note that $r = -1$ is not a solution of the characteristic equation so the particular solution of the equation will have the form

$$y_{p_3}(x) = \alpha e^{-x}$$

where α is a real constant to be determined. We derive twice y_{p_2}

$$y'_{p_3} = -\alpha e^{-x},$$
$$y''_{p_3} = \alpha e^{x},$$

and we replace in the equation, from which

$$\alpha = \frac{5}{8}$$

therefore the particular solution of the equation takes the form

$$y_{p_3}(x) = \frac{5}{8}e^{-x}$$

From which the particular solution of the equation

$$y_p(x) = (x+2) - 2xe^x + \frac{5}{8}e^{-x}$$

and thus the general solution of nonhomogeneous equation is

$$y_{gle}(x) = Ae^x + Be^{3x} + (x+2) - 2xe^x + \frac{5}{8}e^{-x}, \quad A, B \in \mathbb{R}$$

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